CIS 5800

Machine Perception

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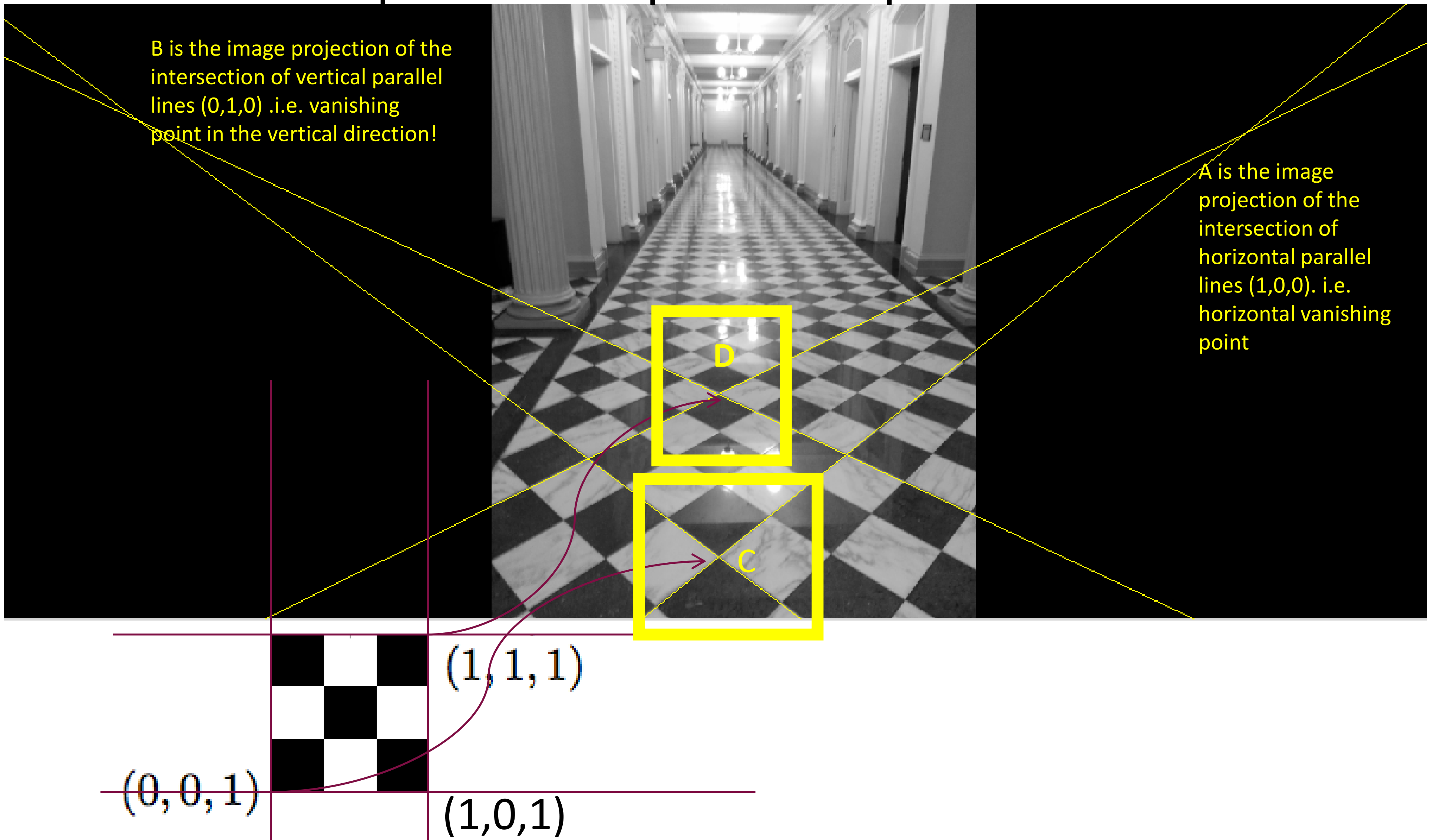
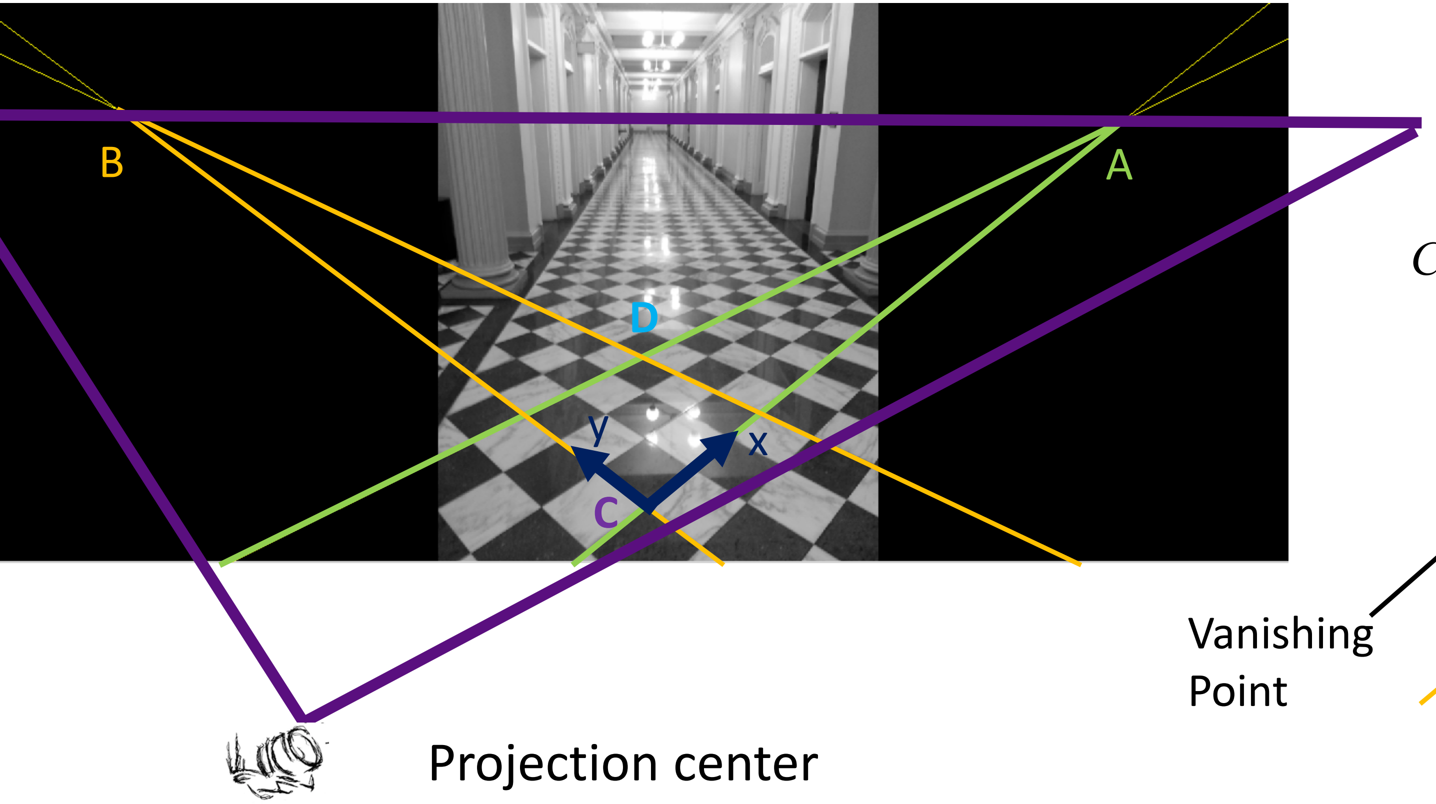
Lec 6: Feb 12, 2025

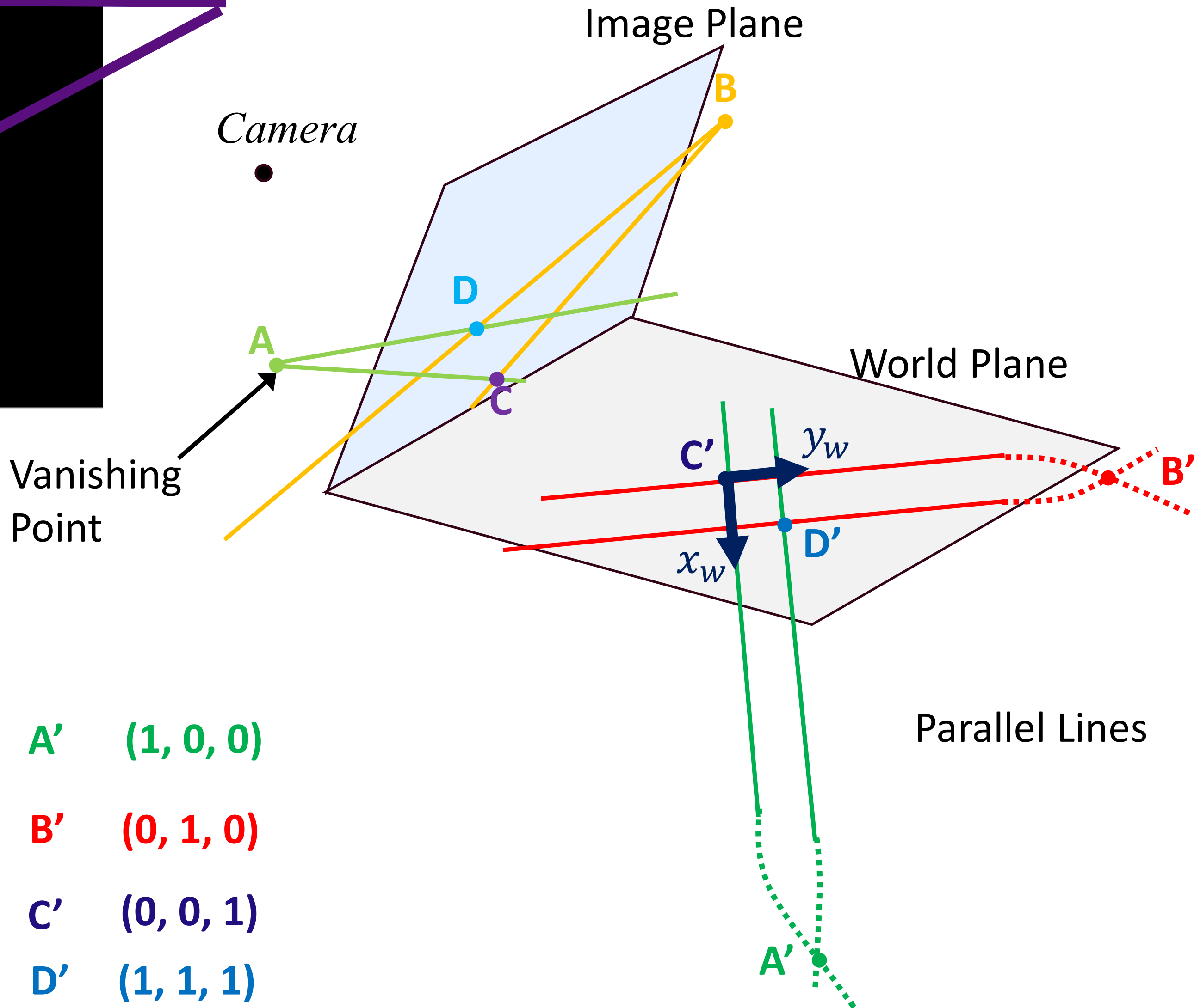
Recap: Computing Homography from 4 Point Correspondences

Note: makes sense, because after all, H has 8 degrees of freedom, and each 2D point correspondence pins down 2DOF.



Recap: Computing Homography from 4 Point Correspondences





Correspondences:
($P^2 \rightarrow P^2$)

A	a	⟷	A’	(1, 0, 0)
B	b	⟷	B’	(0, 1, 0)
C	c	⟷	C’	(0, 0, 1)
D	d	⟷	D’	(1, 1, 1)

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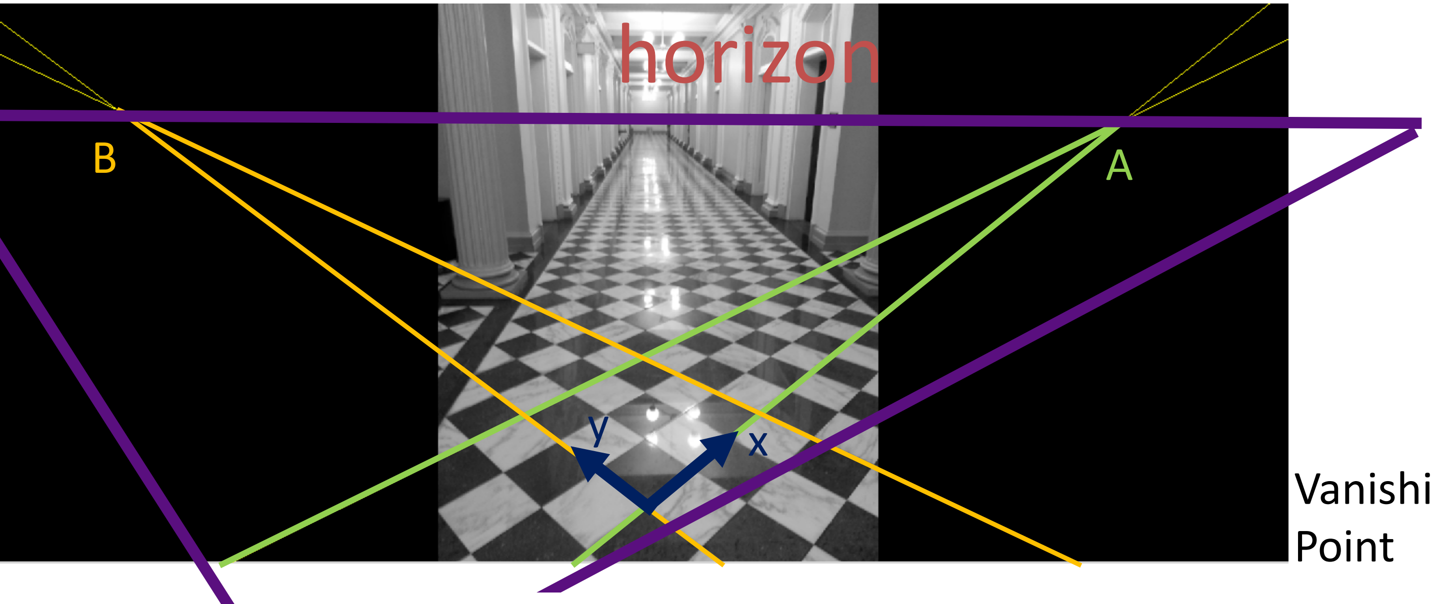
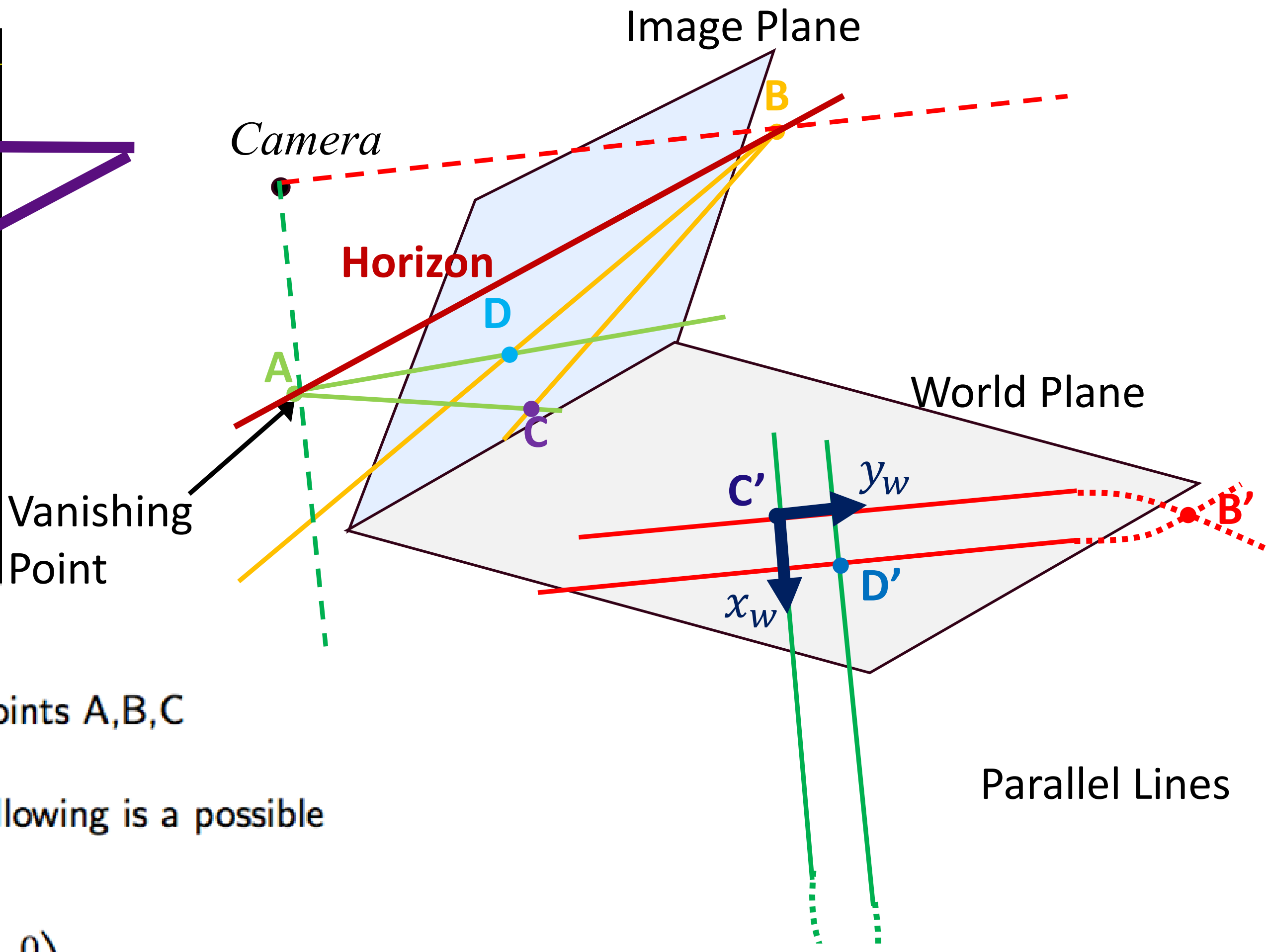




Assume that a mapping A maps the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to the non-collinear points A,B,C

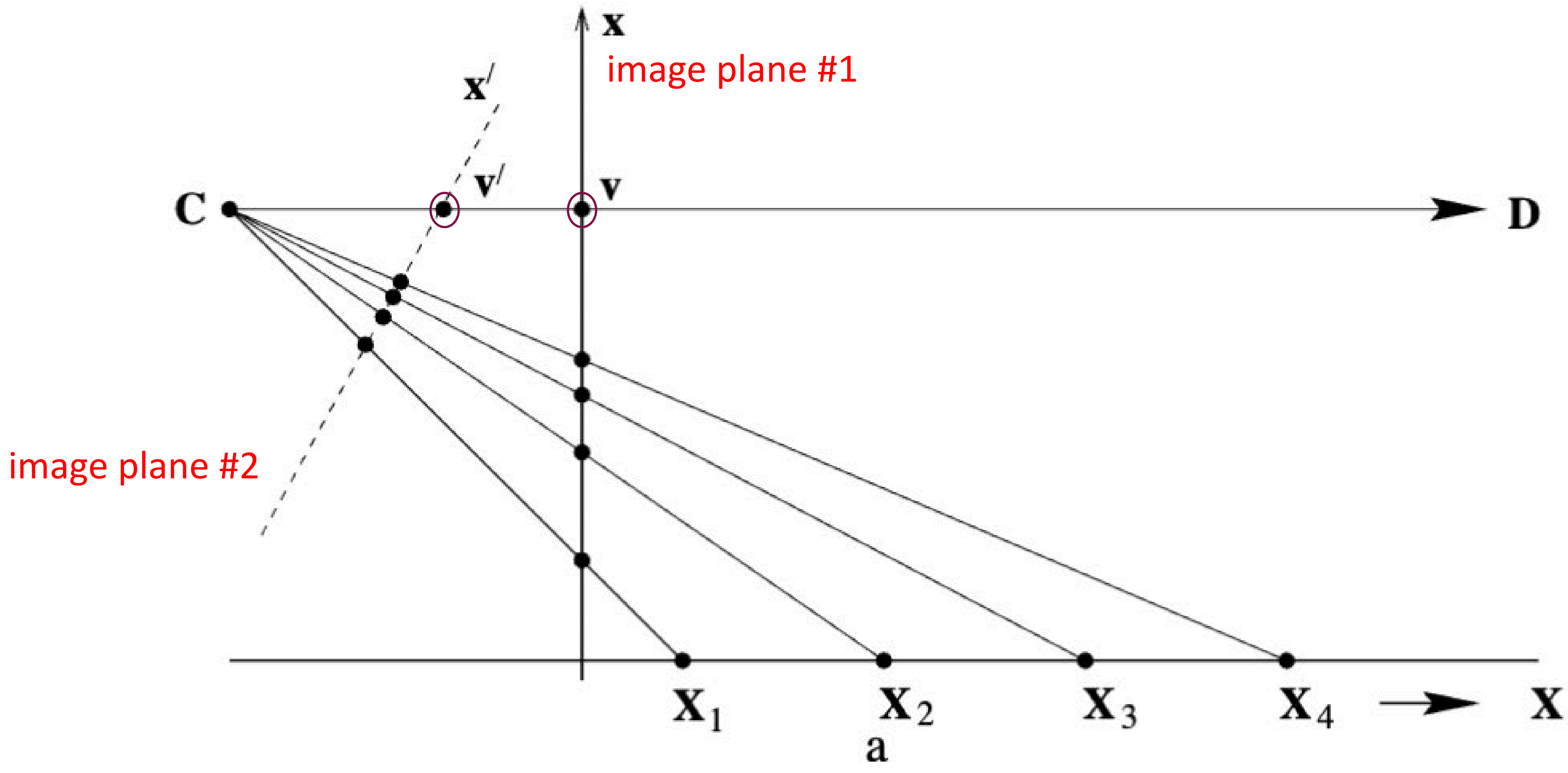
with coordinate vectors a, b and $c \in \mathbb{P}^2$. Then the following is a possible projective transformation:

$$\begin{pmatrix} a & b & c \end{pmatrix} = \underbrace{\begin{pmatrix} \boxed{\alpha a} & \boxed{\beta b} & \gamma c \end{pmatrix}}_{H_{3\times 3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $H = \begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix}$ then $h_1 \sim a$ and $h_2 \sim b$

The first two columns of Homography are two orthogonal vanishing points

Recap: The line connecting the camera origin and the vanishing point is parallel to all lines that share the same direction and converge at the vanishing point.

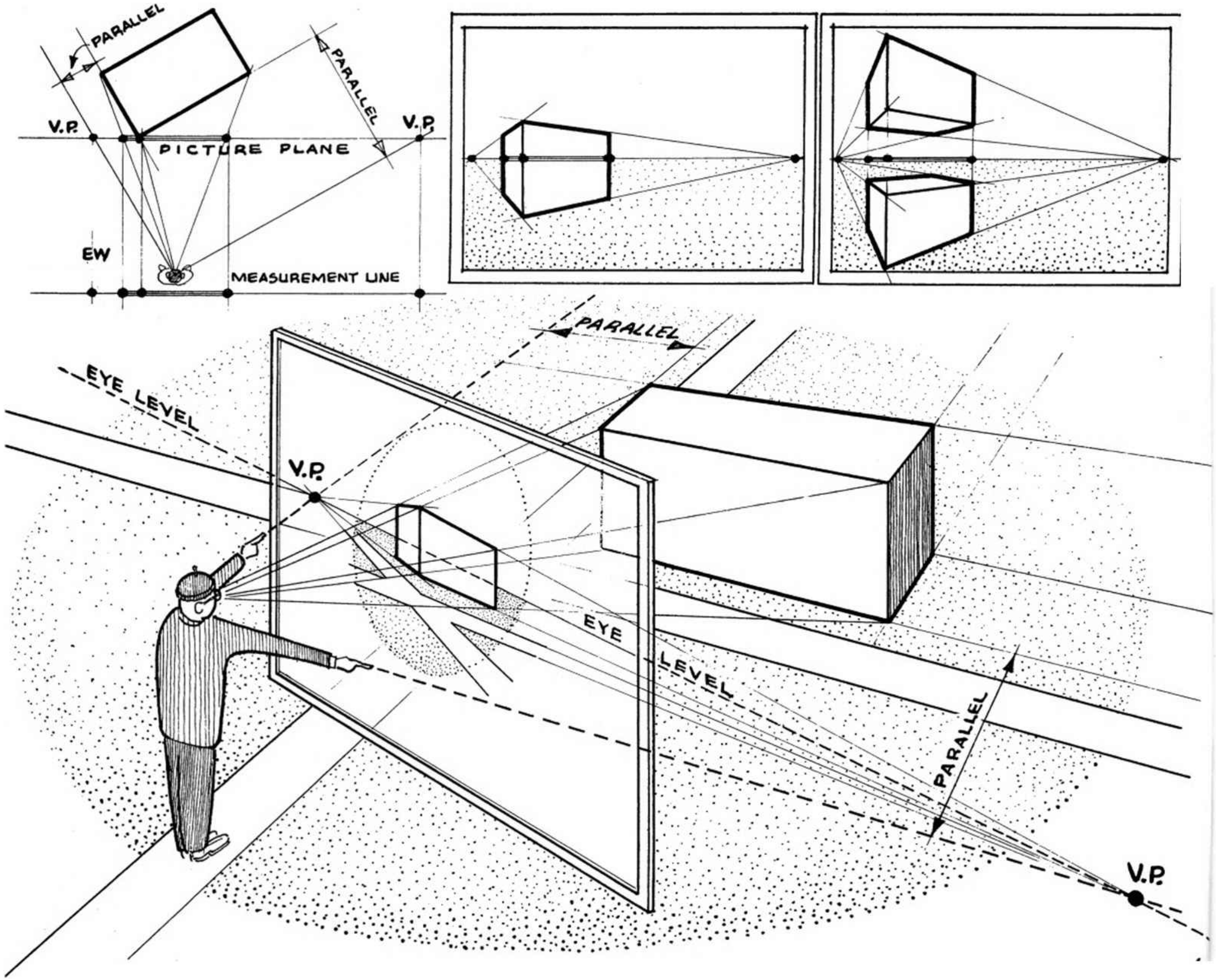


Recap: How Artists Find Vanishing Points

Find VP of a world line by:

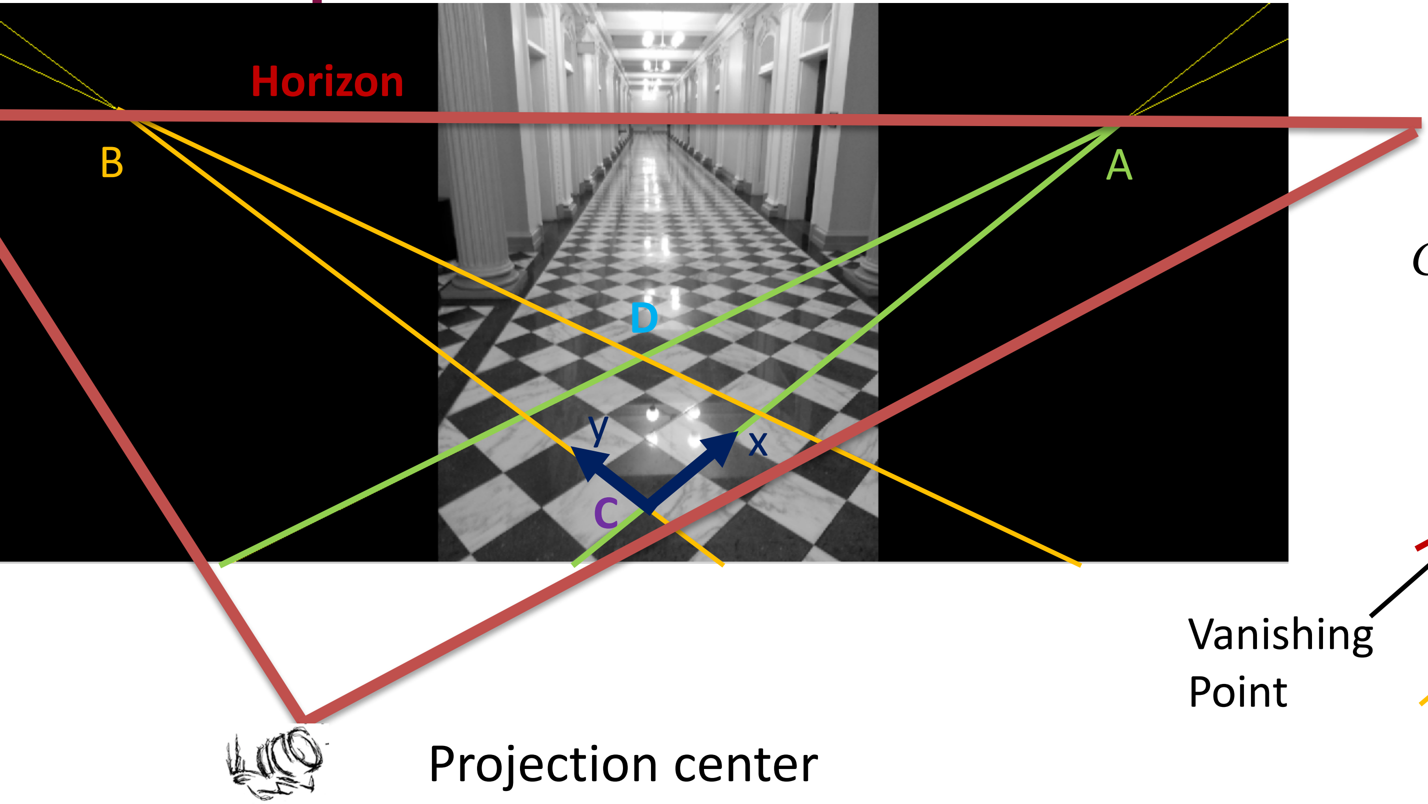
- Standing at “camera center”.
- Holding arm out parallel to the world line.
- Noting its intersection with the ”canvas” or image plane. i.e. the arm represents the light ray.

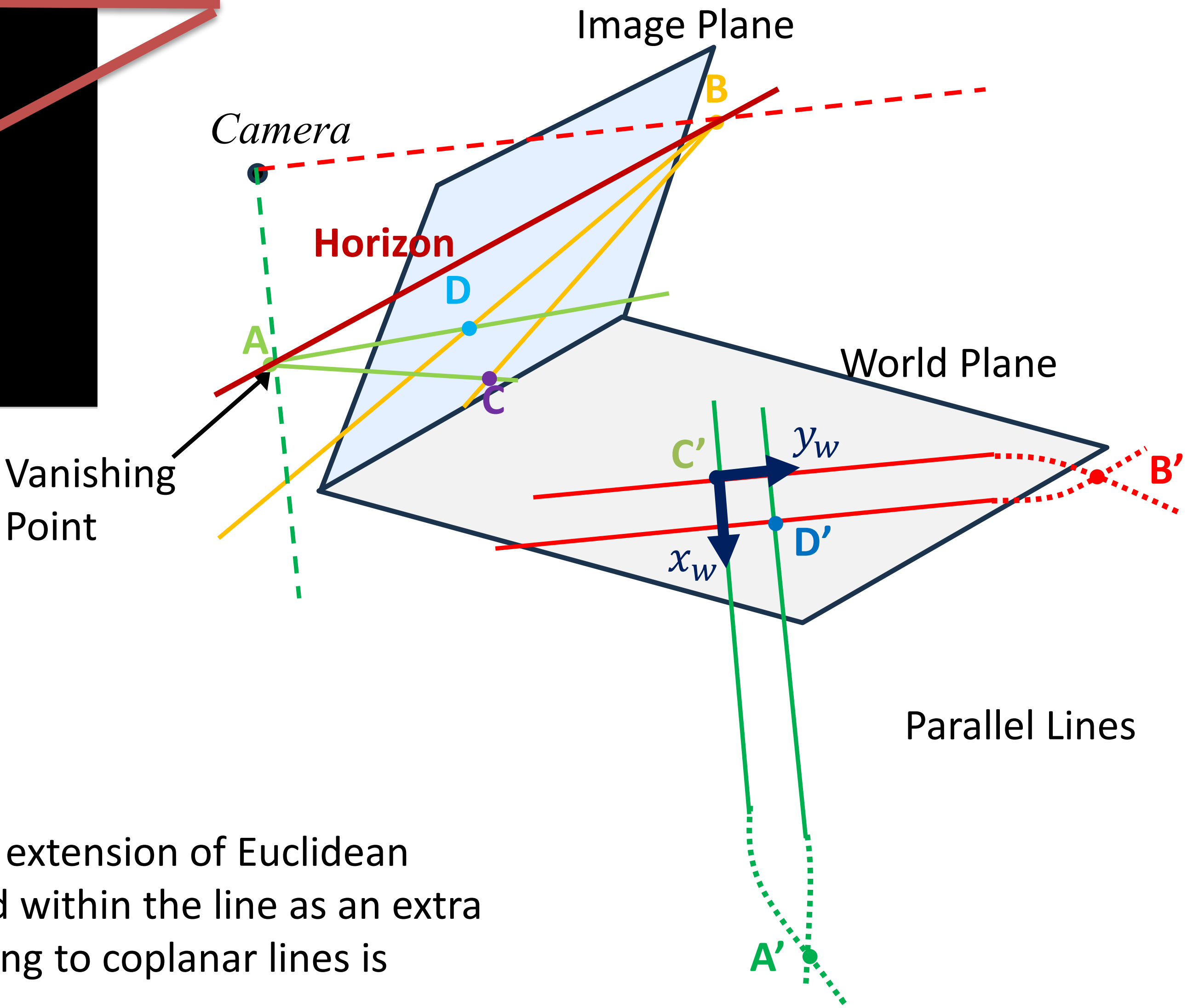
“Vanishing rays of a world line” (camera rays through the VP) are just rays parallel to that line, passing through the camera center.



http://www.joshuanava.biz/perspective/in-other-words-the-observer-simply-points-in-the-same-direction-as-the-lines-in-order-to-find-their-vanishing-point.html

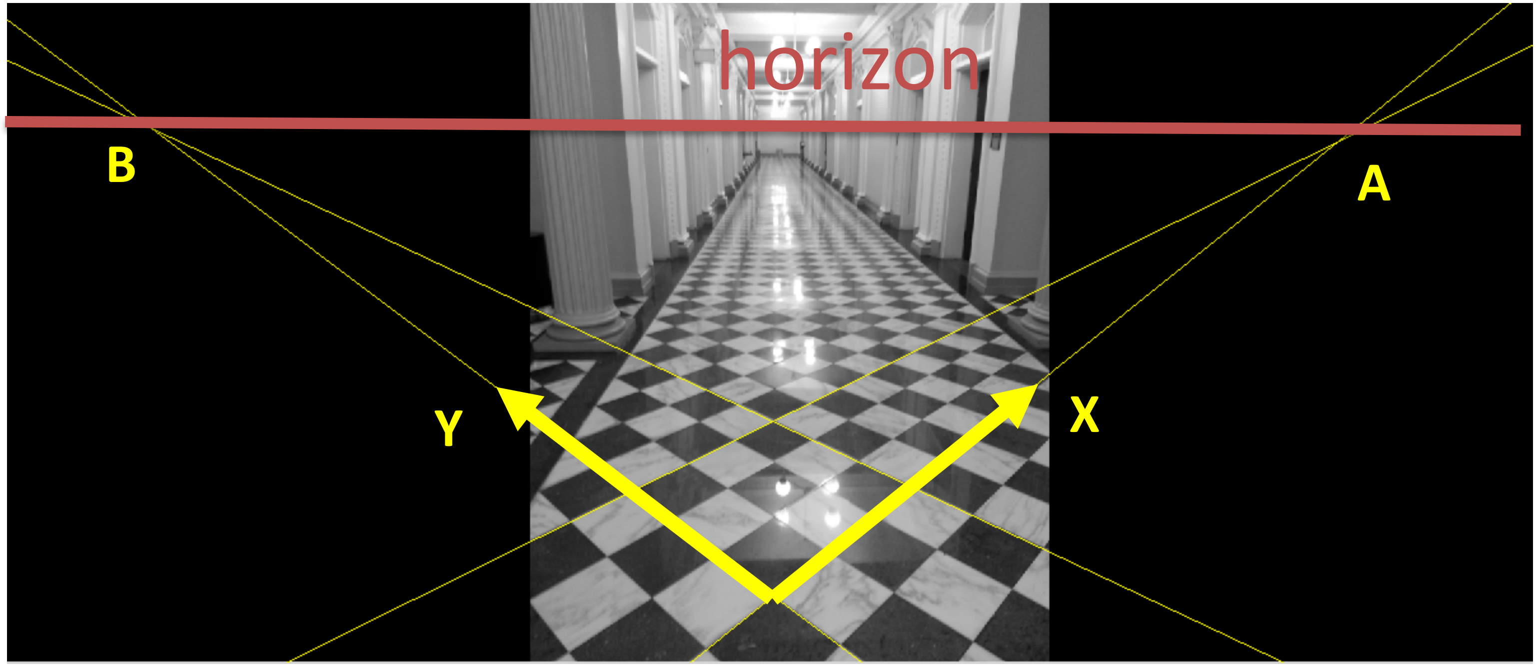
Recap: Horizon





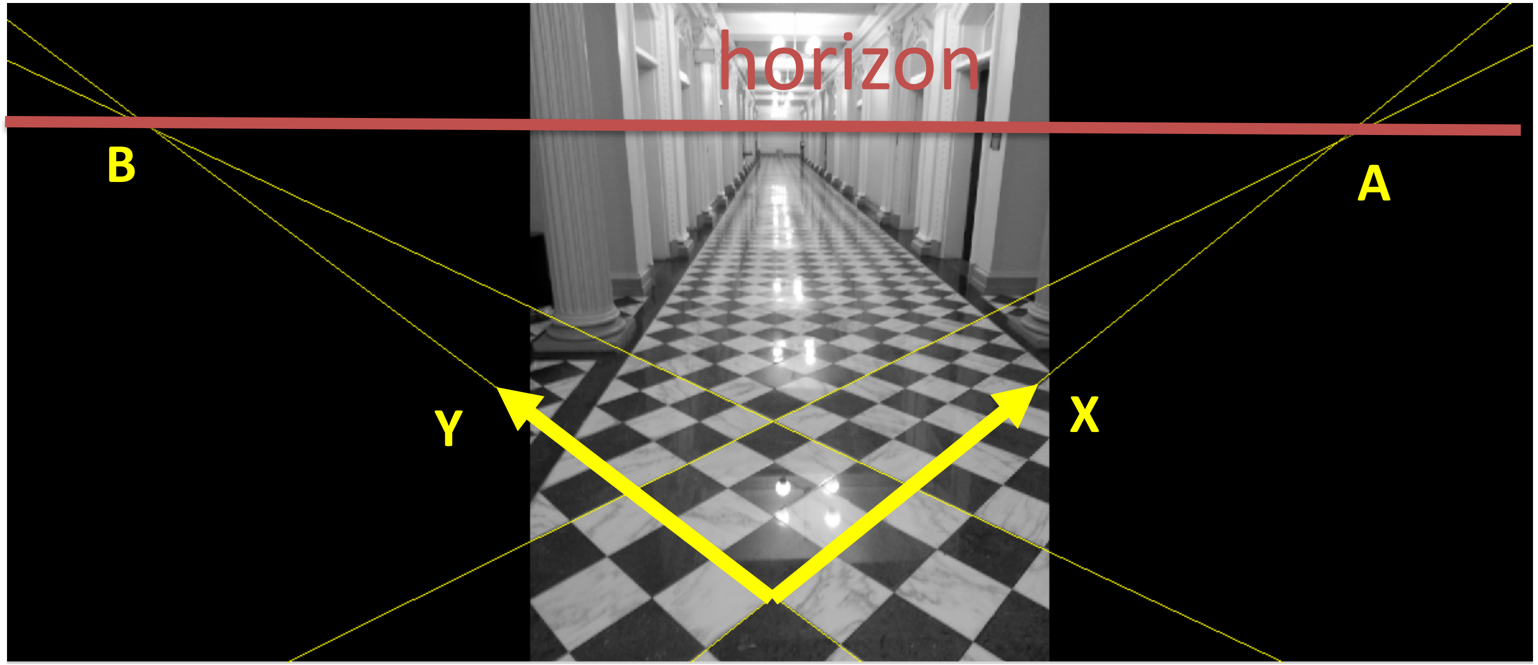
In essence, a projective geometry may be thought of as an extension of Euclidean geometry in which the "direction" of each line is subsumed within the line as an extra "point", and in which a "horizon" of directions corresponding to coplanar lines is regarded as a "line"

If we connect two vanishing points, we obtain the “horizon”!



Side note: with our assumption of the world plane as being the “XY” plane, and following the common convention that xy plane is horizontal, and z is vertical, this indeed maps to our normal notions of a “horizon”.

Equation of horizon: $(h_1 \times h_2)^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

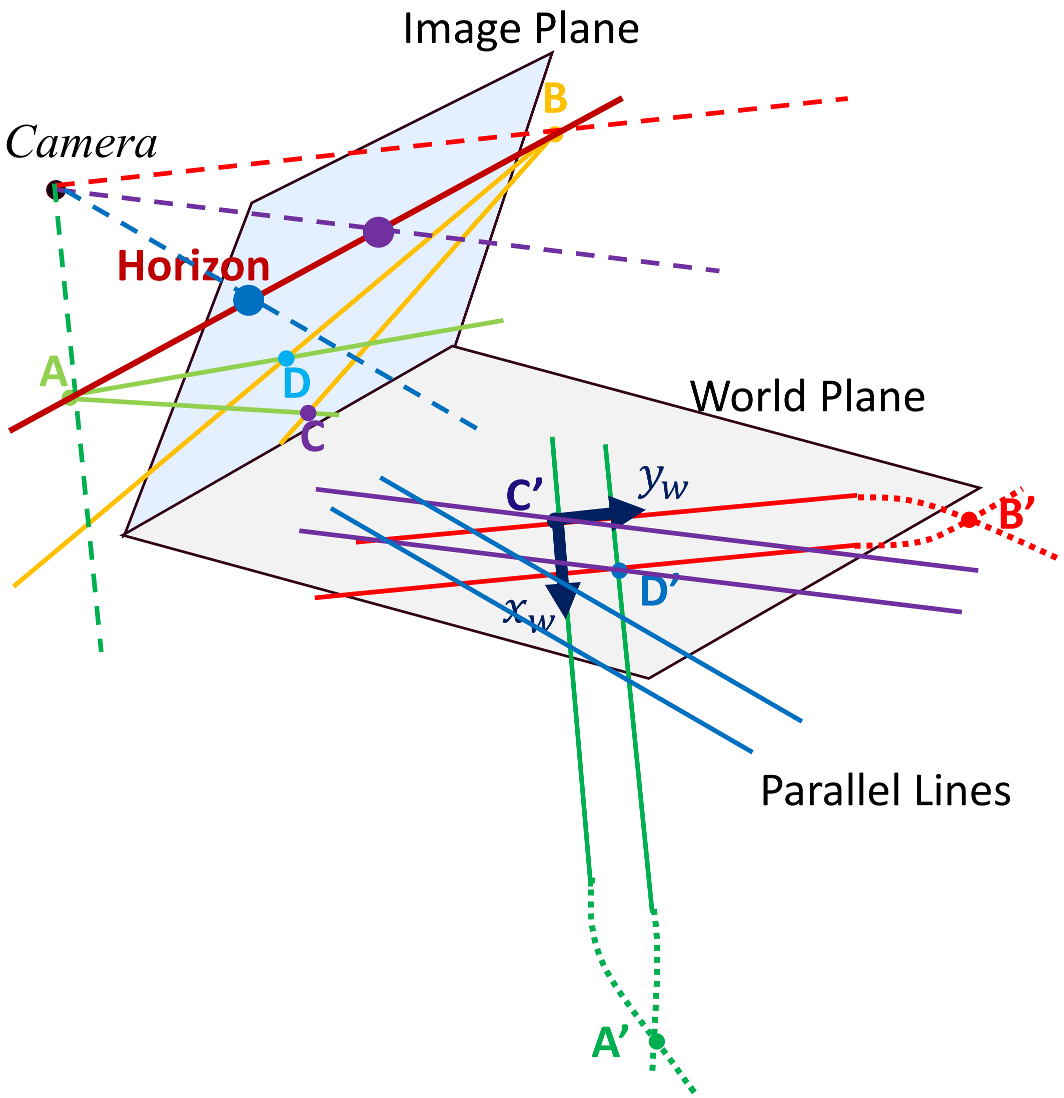


We will encounter another way to derive this equation of the horizon very soon.

Projecting the line at infinity to compute the horizon

Points at infinity in the world plane look like $(X, Y, W = 0)^T$

The "line" connecting them is $W = 0$, the "line at infinity". The image of this line is the horizon, which contains all vanishing points. Expressed in world plane $\mathbb{P}^2$, this line's coefficients are $(0,0,1)^T$.



So if we could find the projection of this line, we could find the horizon

Deriving the equation of a horizon in another way

Q: We know planar projections H transform points $p \in \mathbb{P}^2$ as $p \to Hp$. How do they transform lines in $\mathbb{P}^2$?

Projective Transformation of Lines

If H maps a point to Hp, then where does a line l map to?

Line equation in original plane

$$l^T p = 0$$

Line equation in image plane where any point $p' = Hp$

$$l^T H^{-1} p' = 0$$

Implies that $l' = H^{-T} l$

Projecting the line at infinity to compute the horizon

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The "line" connecting them is $W = 0$, the "line at infinity". The image of this line is the horizon, which contains all vanishing points. Expressed in world plane $\mathbb{P}^2$, this line's coefficients are $(0,0,1)^T$.

So if we could project this line, we could find the horizon

We have just seen projections of lines are $H^{-T} l$, so the horizon is $H^{-T}(0,0,1)^T$.

If $H = [h_1\ h_2\ h_3]$ then H^{-T} is $[h_2 \times h_3\ h_3 \times h_1\ h_1 \times h_2]$, so the horizon line $H^{-T}(0,0,1)^{\mathrm{T}} = h_1 \times h_2$.

This is consistent: the horizon connects the two vanishing points h_1 and h_2.

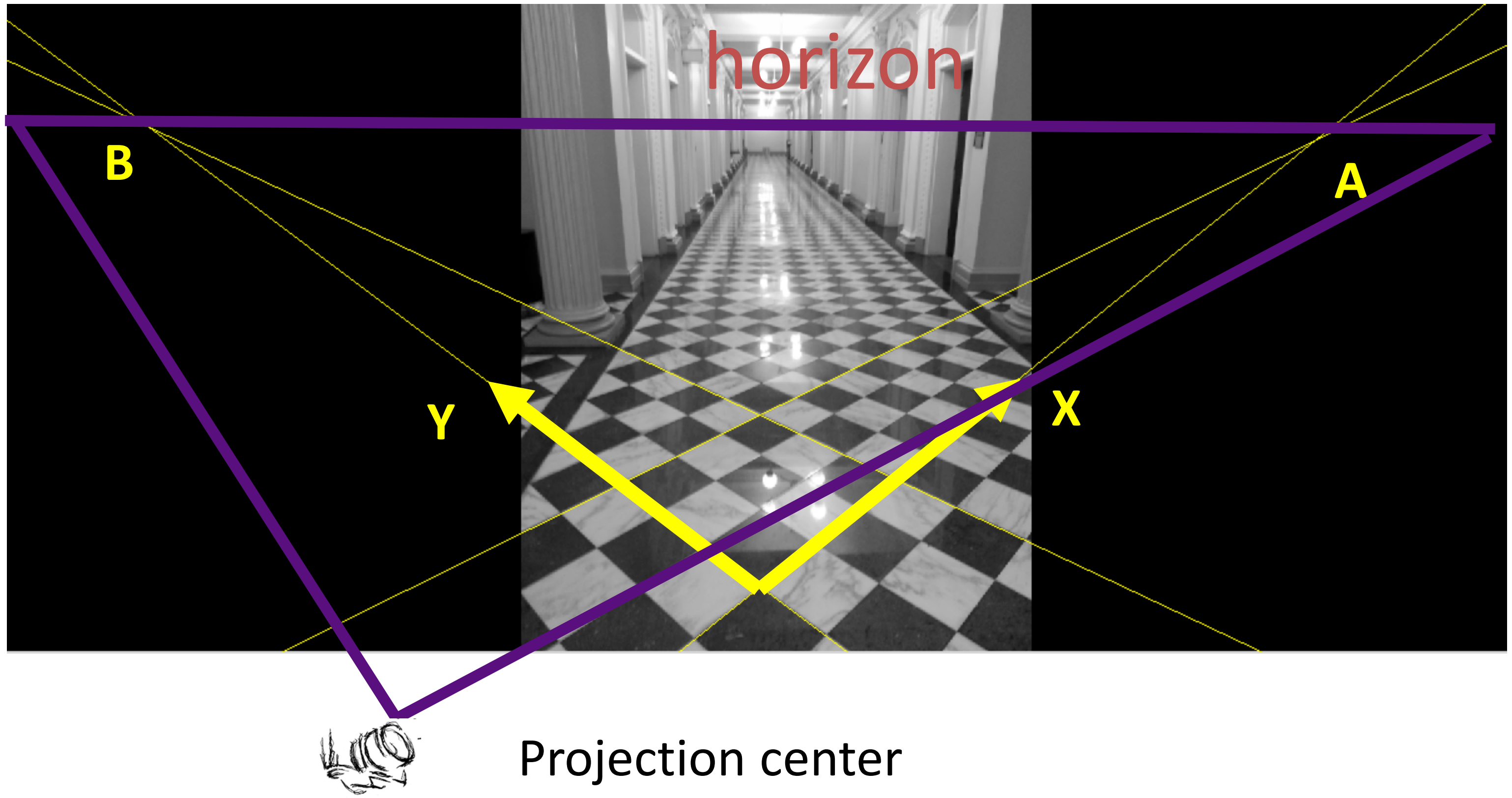
Summary

Equation of horizon is $h_1 \times h_2$

$$=$$

Normal to the viewing plane through horizon is $h_1 \times h_2$

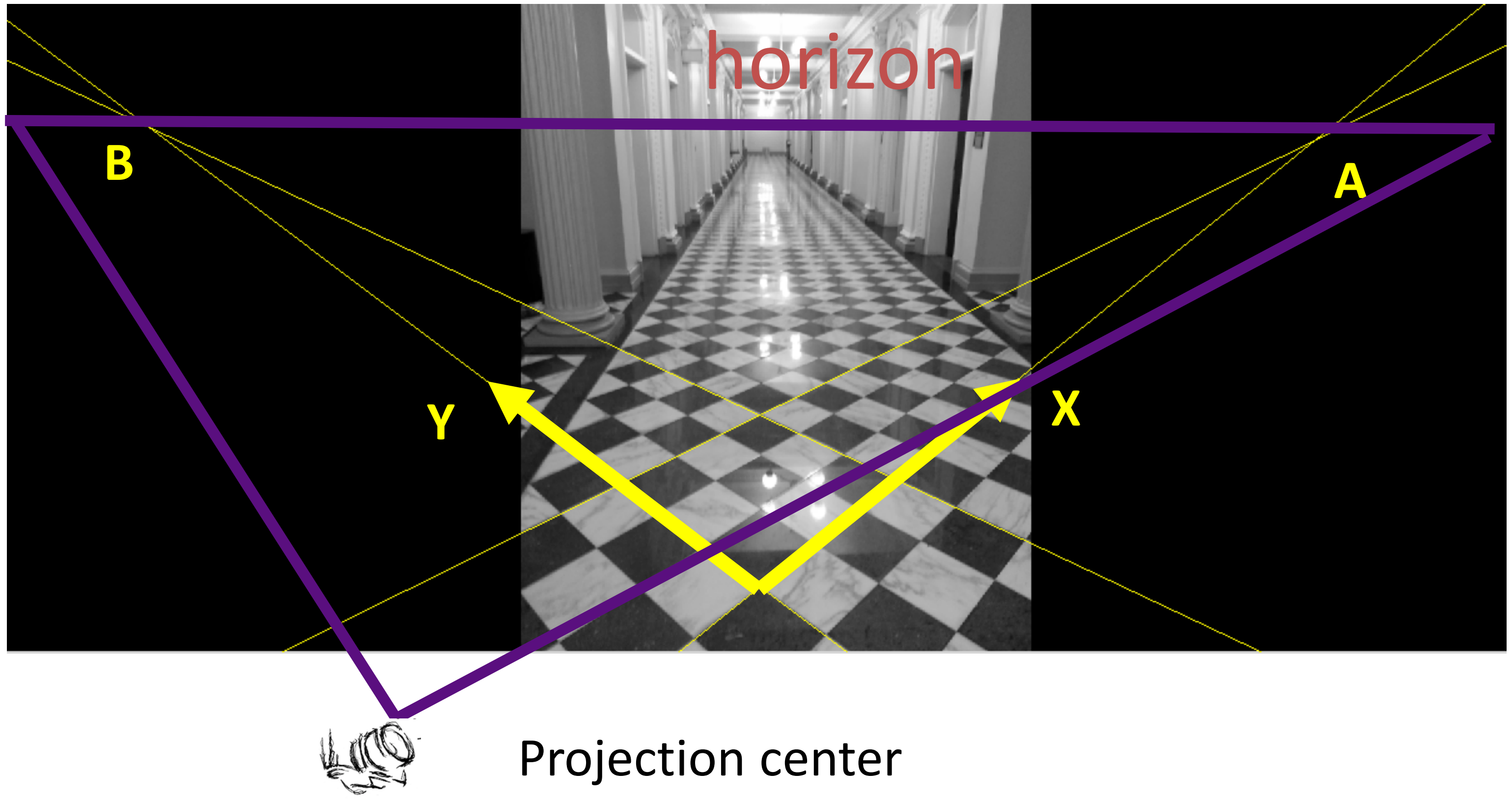
Plane through camera center and horizon line on image plane = vanishing plane of the world plane



Summary

Vanishing rays/planes through the camera center are parallel to the world lines/planes

So, the horizon plane is parallel to the ground plane
and hence $h_1 \times h_2$ is the normal to the ground plane!



Horizon plane =
Vanishing plane =
Viewing plane

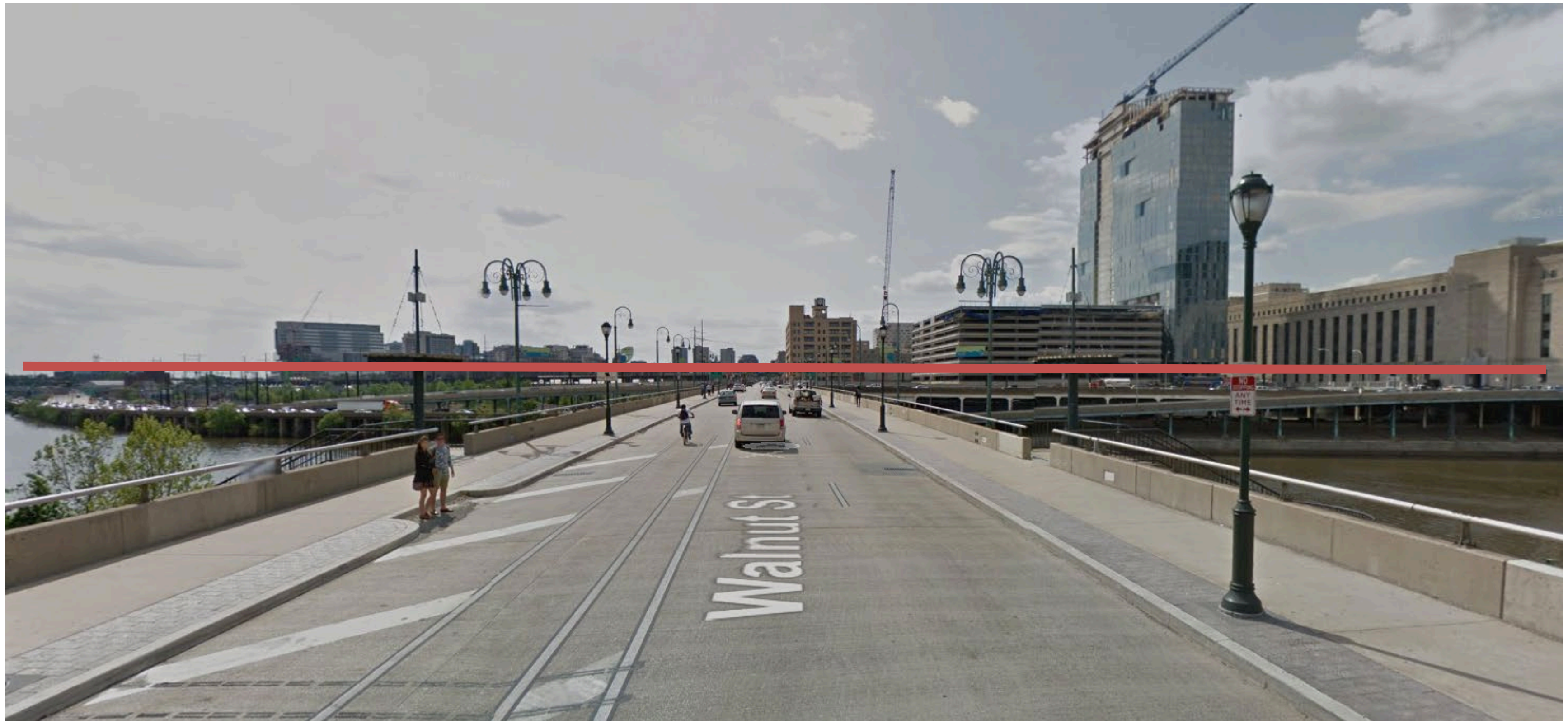
World plane //
vanishing plane

World plane = Ground
plane in this case

Using the horizon to orient the camera

Horizon gives complete info about how ground plane is oriented*!

Thumb rule: “If horizon is horizontal & central, camera is correctly vertical & principal axis is parallel to world plane**!”



*caveat: assuming known K

**caveat: assuming that principal axis passes through image center, and sensor axes are horizontal. (usually approximately true)

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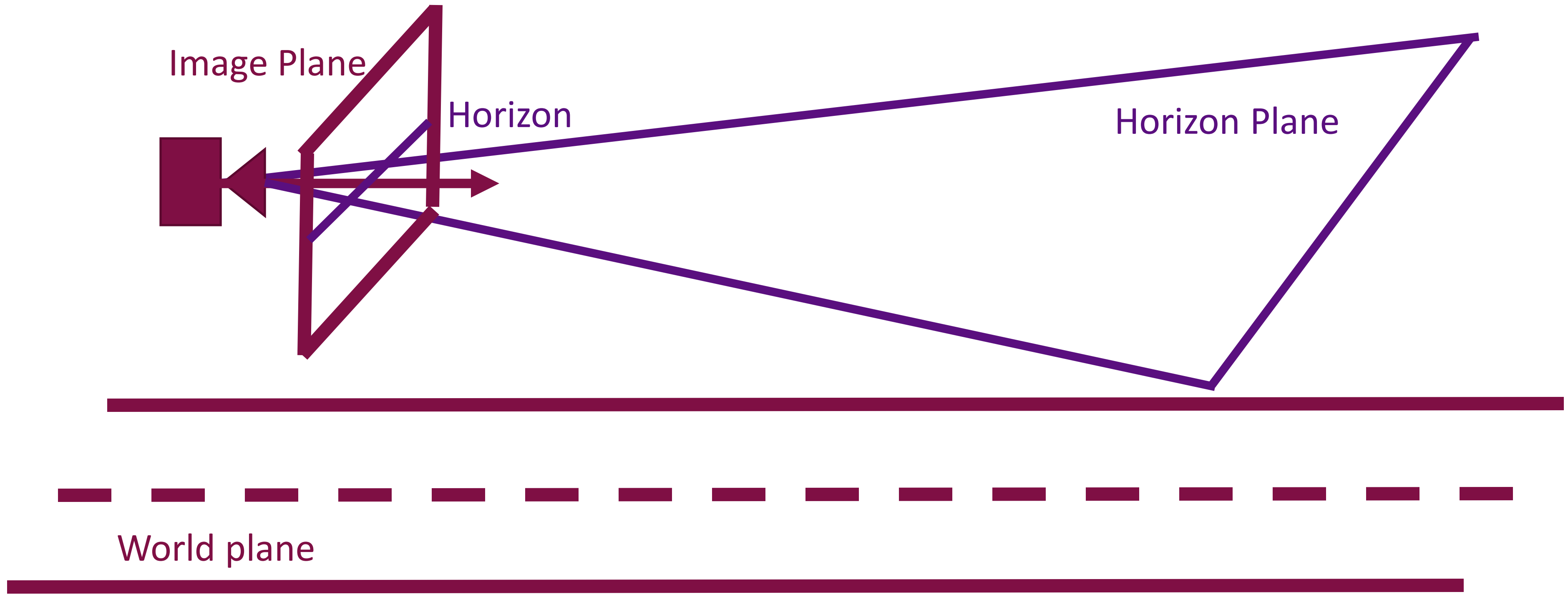
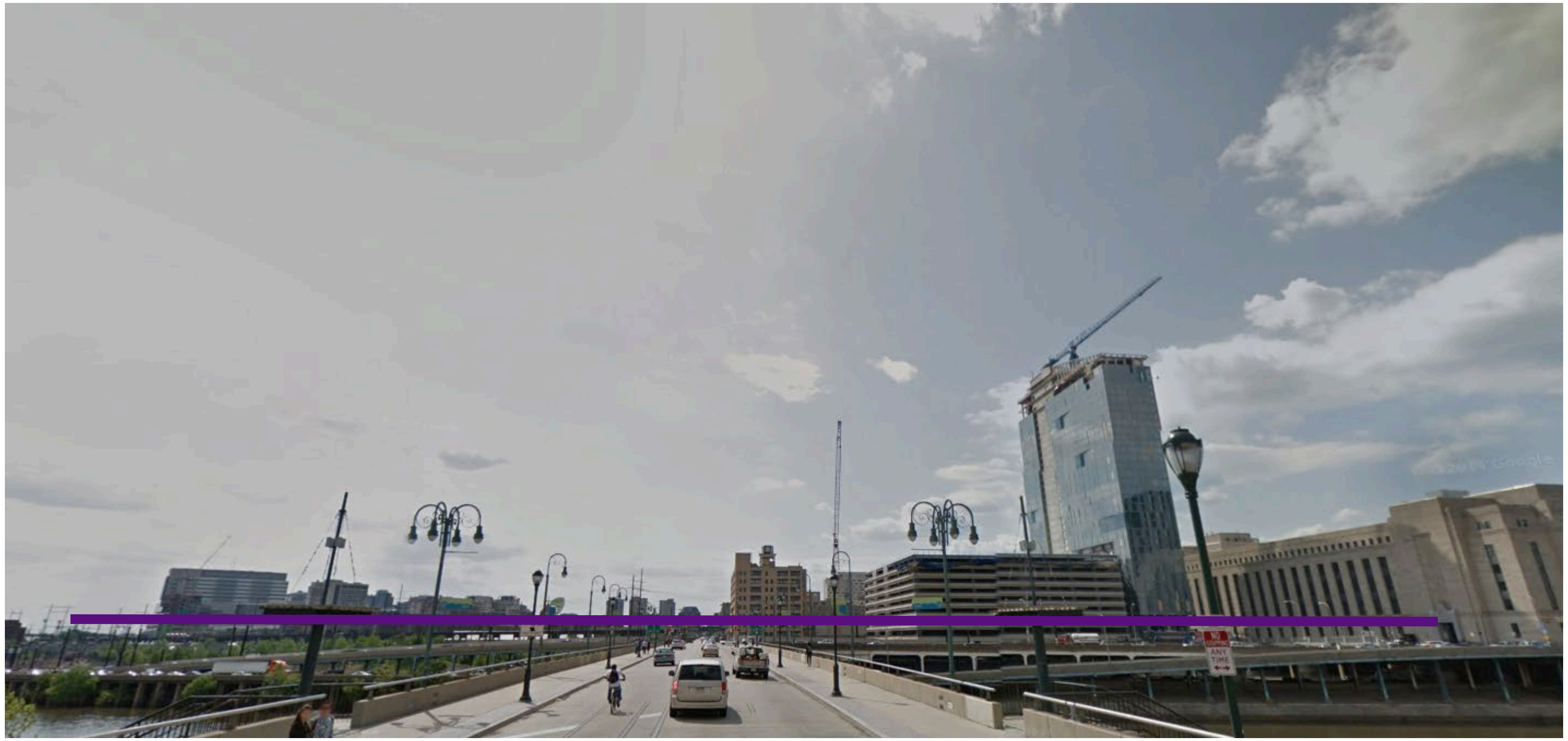
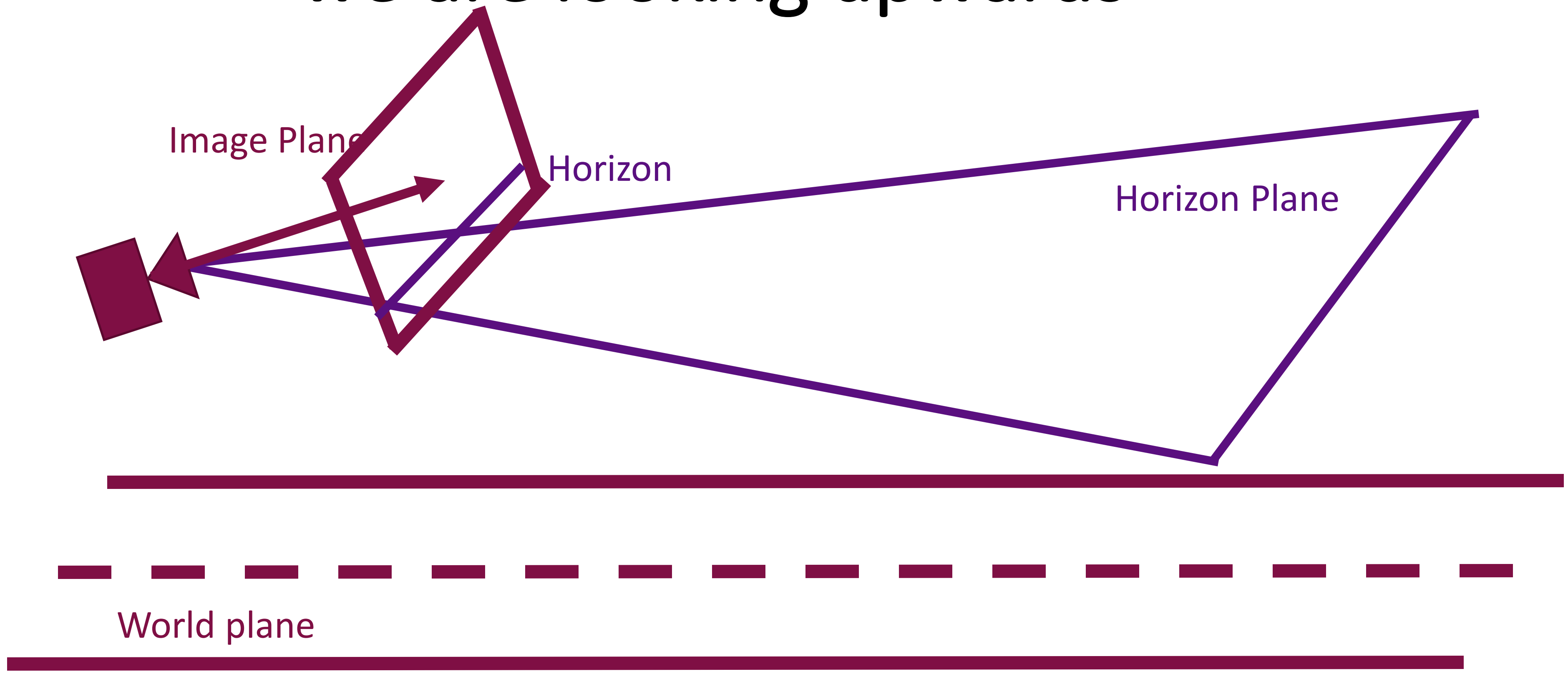
Thumb rule: “If horizon is horizontal & central, camera is correctly vertical & principal axis is parallel to world plane**!”



Horizon below middle of image => we are looking upwards

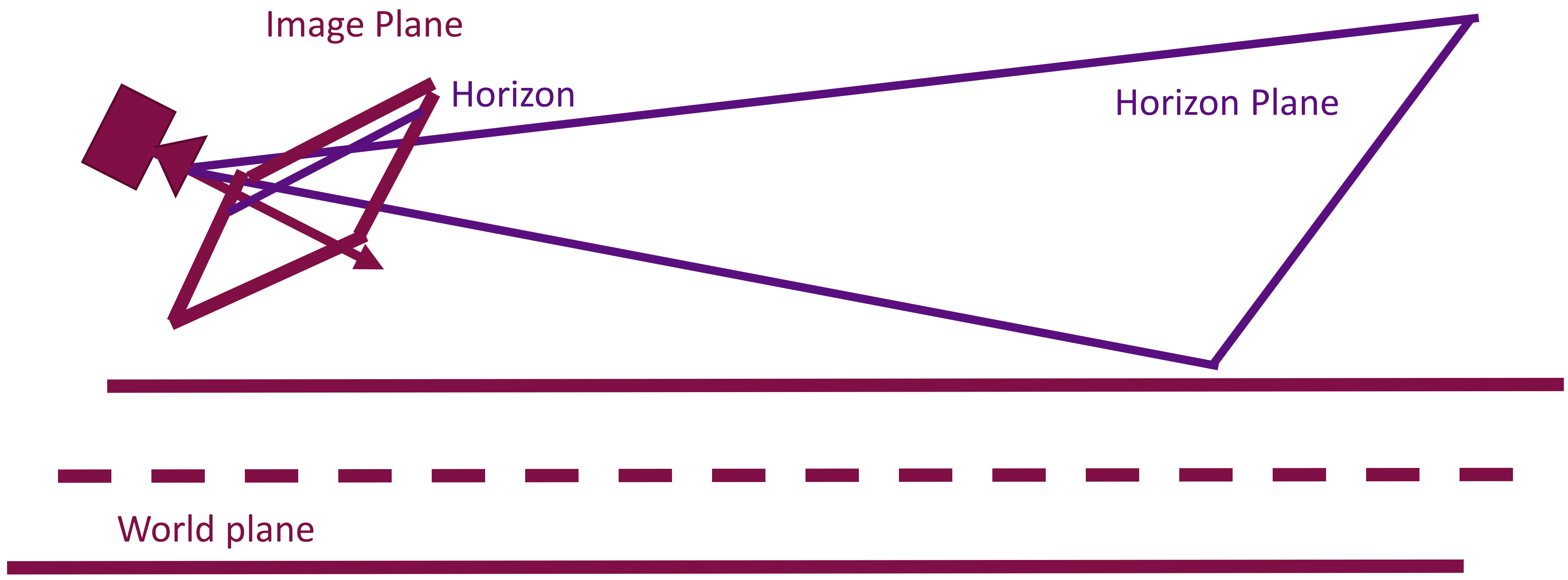


Horizon below middle of image => we are looking upwards

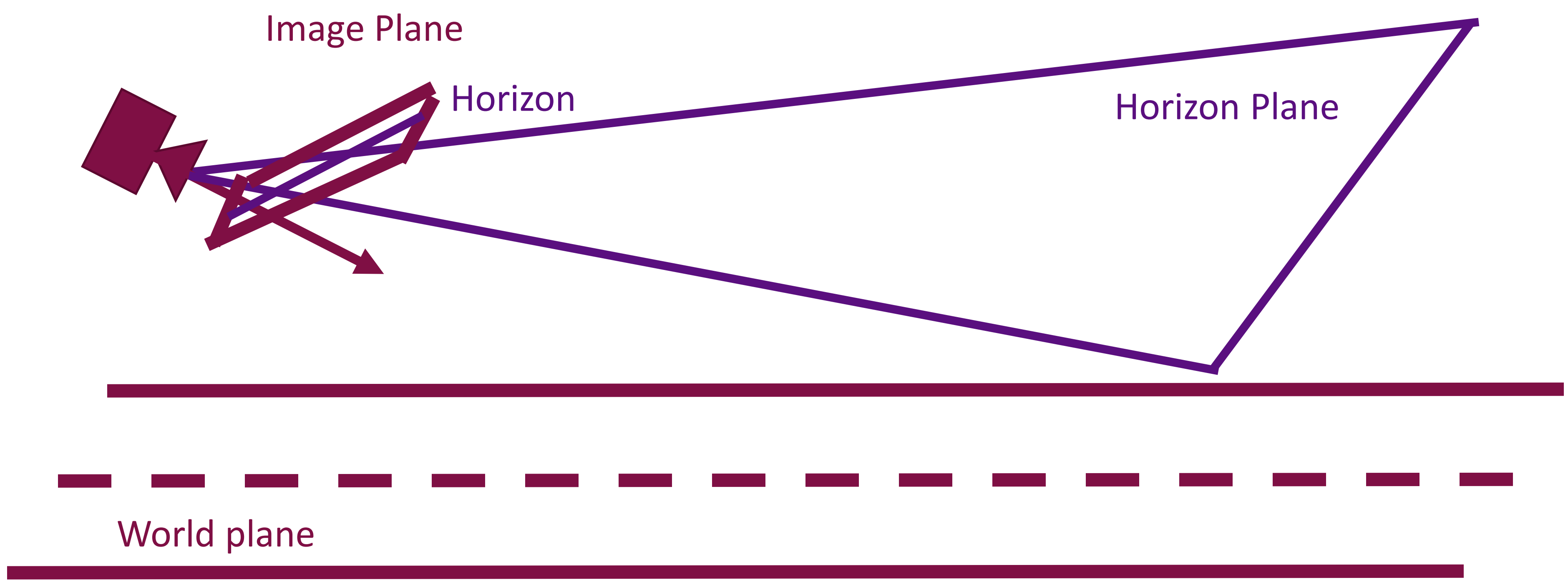


Horizon above middle of image => we are looking downwards

Horizon above middle of image => we are looking downwards



Horizon above middle of image => we are looking downwards

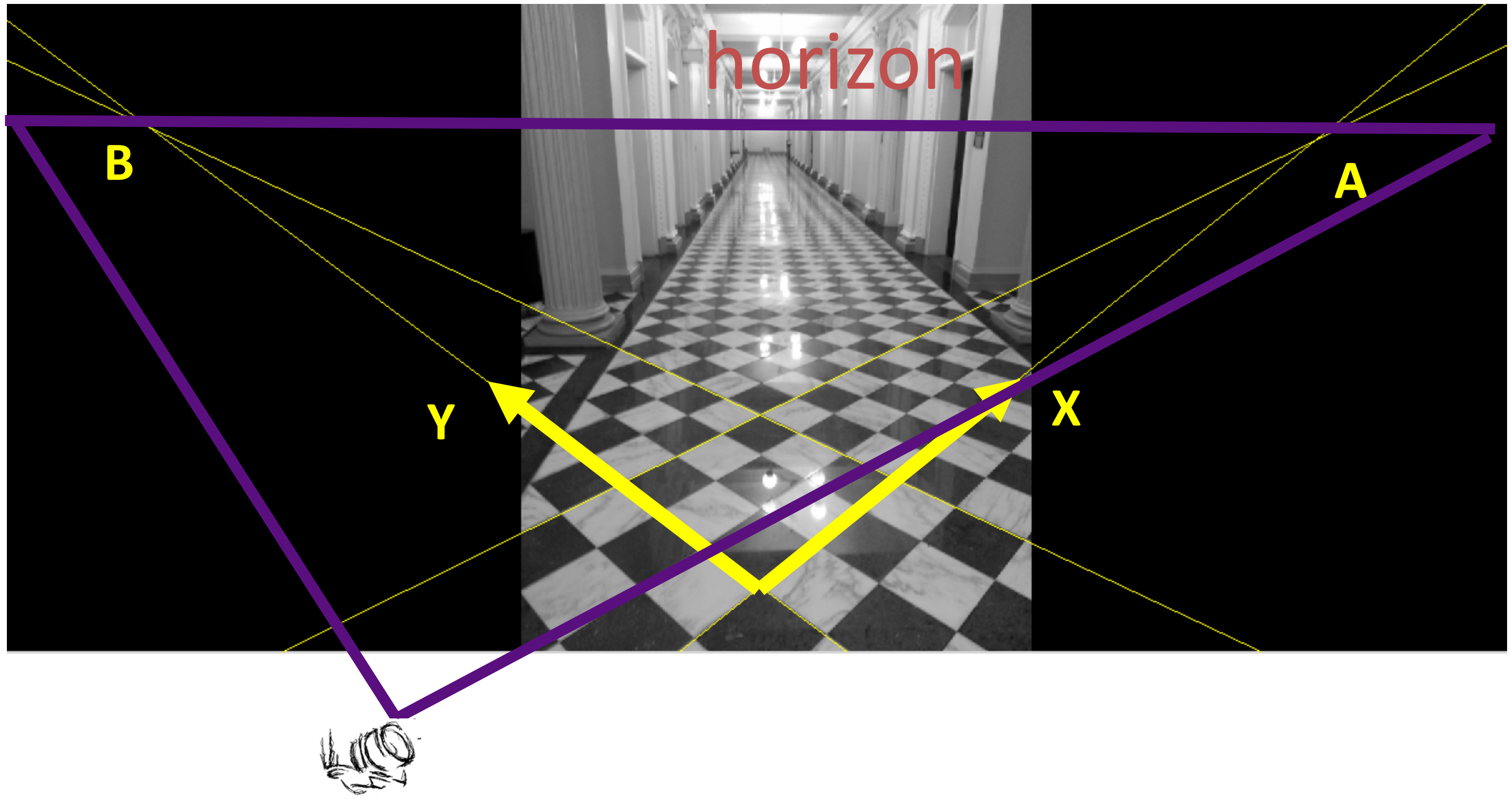


Q: What if you cropped the image? Would these rules still hold?

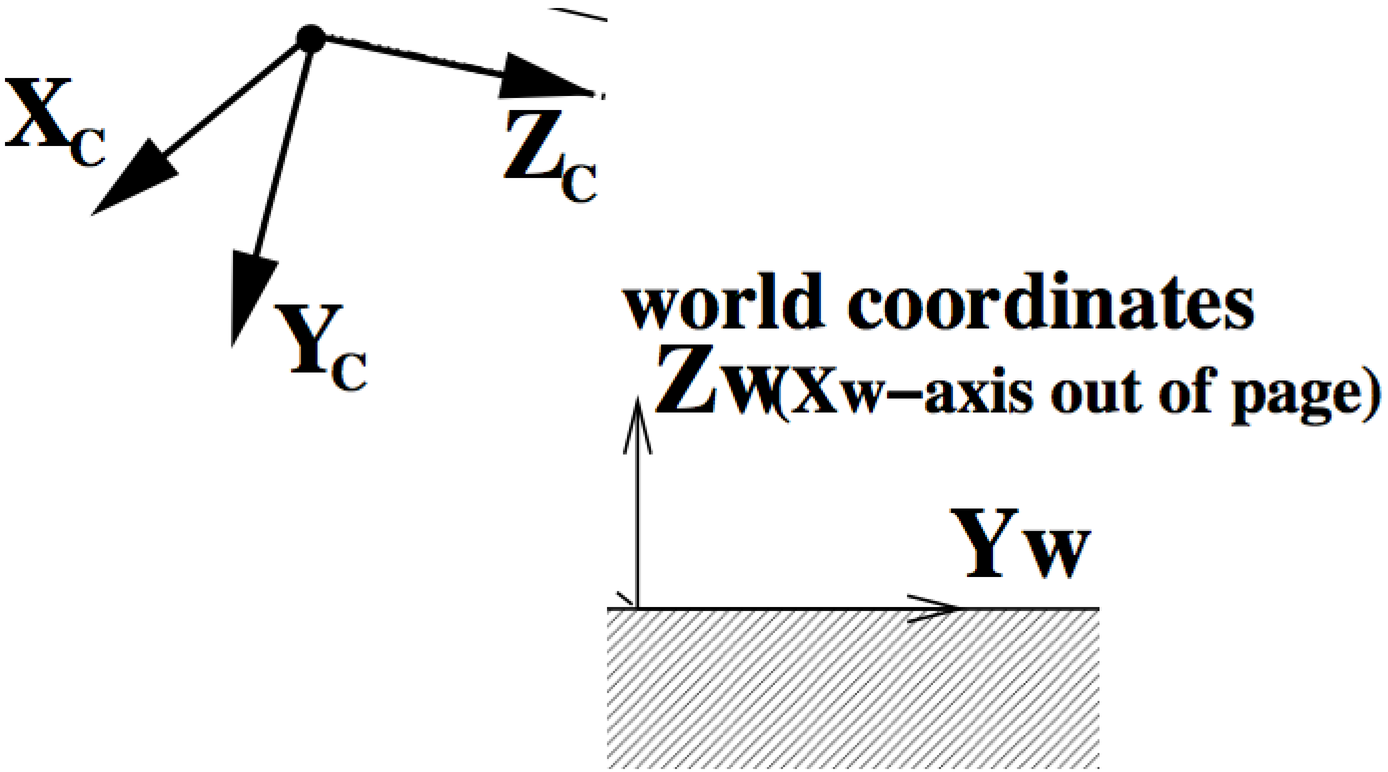
A: No. The effective intrinsics K would change and become “non-standard”, so these thumb rules wouldn’t hold.

Summing up

Horizon tells us how camera is oriented.
Constrains homography.



Recap: Assume world plane $Z_w = 0$



Recap: Pose From Homography

Computing the homography can tell us how the camera (and therefore, e.g. a robot attached to the camera) is oriented w.r.t. to a world plane! (assuming known K)

Q: Where do you get $\boldsymbol{r}_3$ from though? A: $\boldsymbol{r}_3 = \boldsymbol{r}_1 \times \boldsymbol{r}_2$

Recall the projection from world to camera

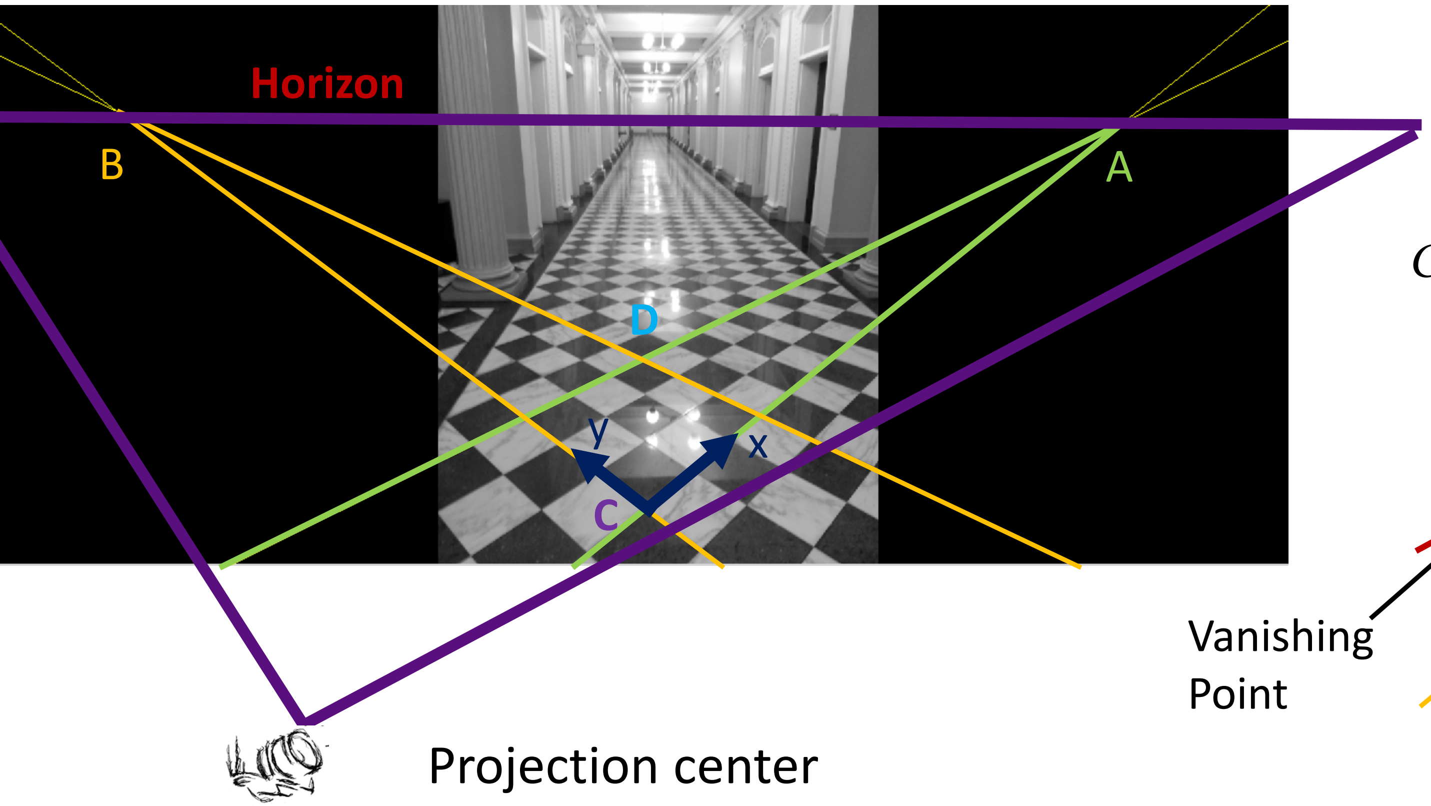
$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim K \begin{pmatrix} r_1 & r_2 & r_3 & T \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}$$

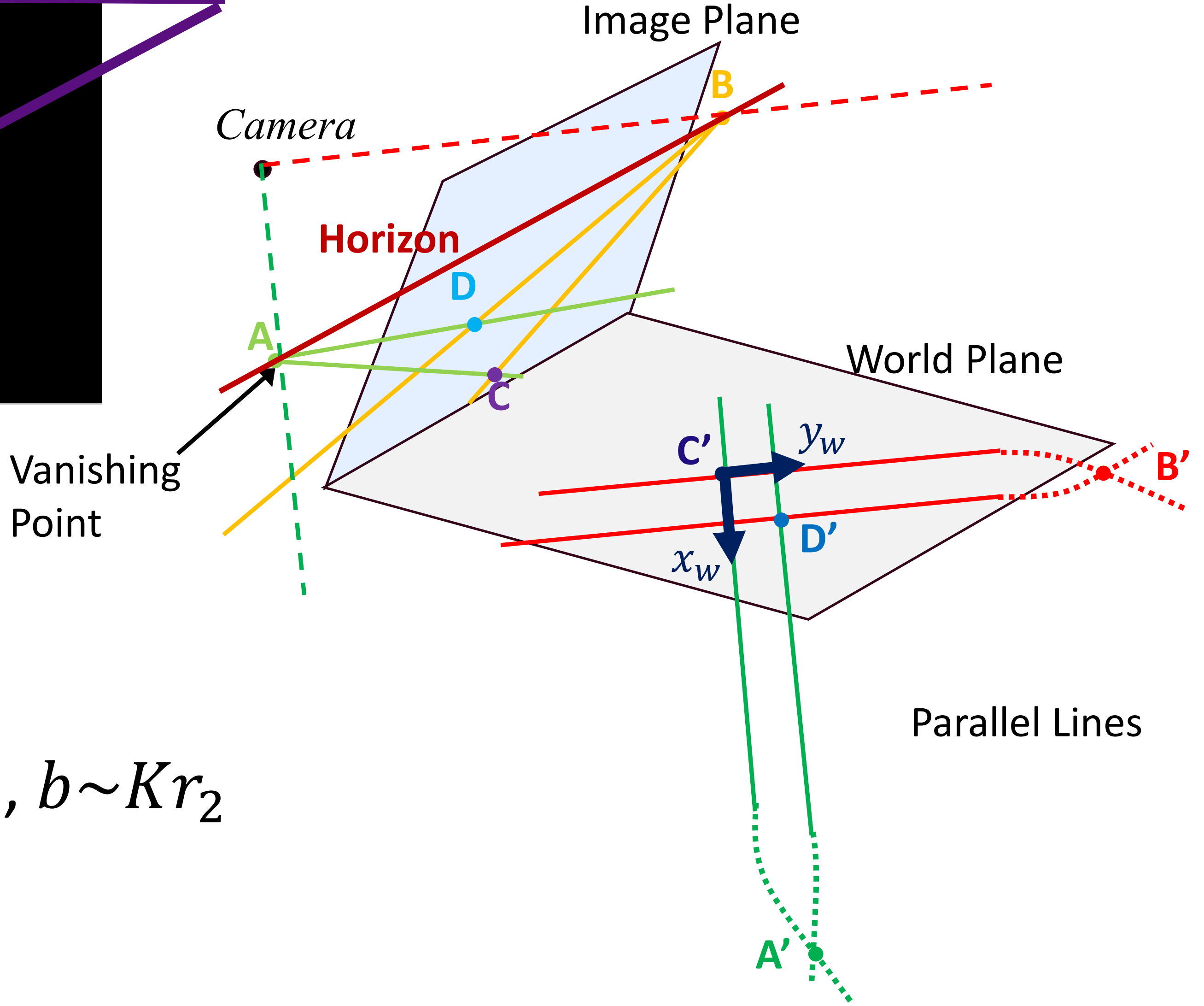
and assume that all points in the world lie in the ground plane $Z = 0$.

Then the transformation reads

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim \underbrace{K \begin{pmatrix} r_1 & r_2 & T \end{pmatrix}}_{} \begin{pmatrix} X \\ Y \\ W \end{pmatrix}$$

The planar homography
$H: \mathbb{P}^2 \rightarrow \mathbb{P}^2$





$$H \sim K(r_1, r_2, T)$$

$$H \sim (a, b, c)$$

$$a \sim K r_1, \; b \sim K r_2$$

- We can get the rotation of camera from 2 orthogonal vanishing points on a plane, **assuming known intrinsics matrix** K.

- Turns out, you can also get *intrinsics* from vanishing points.

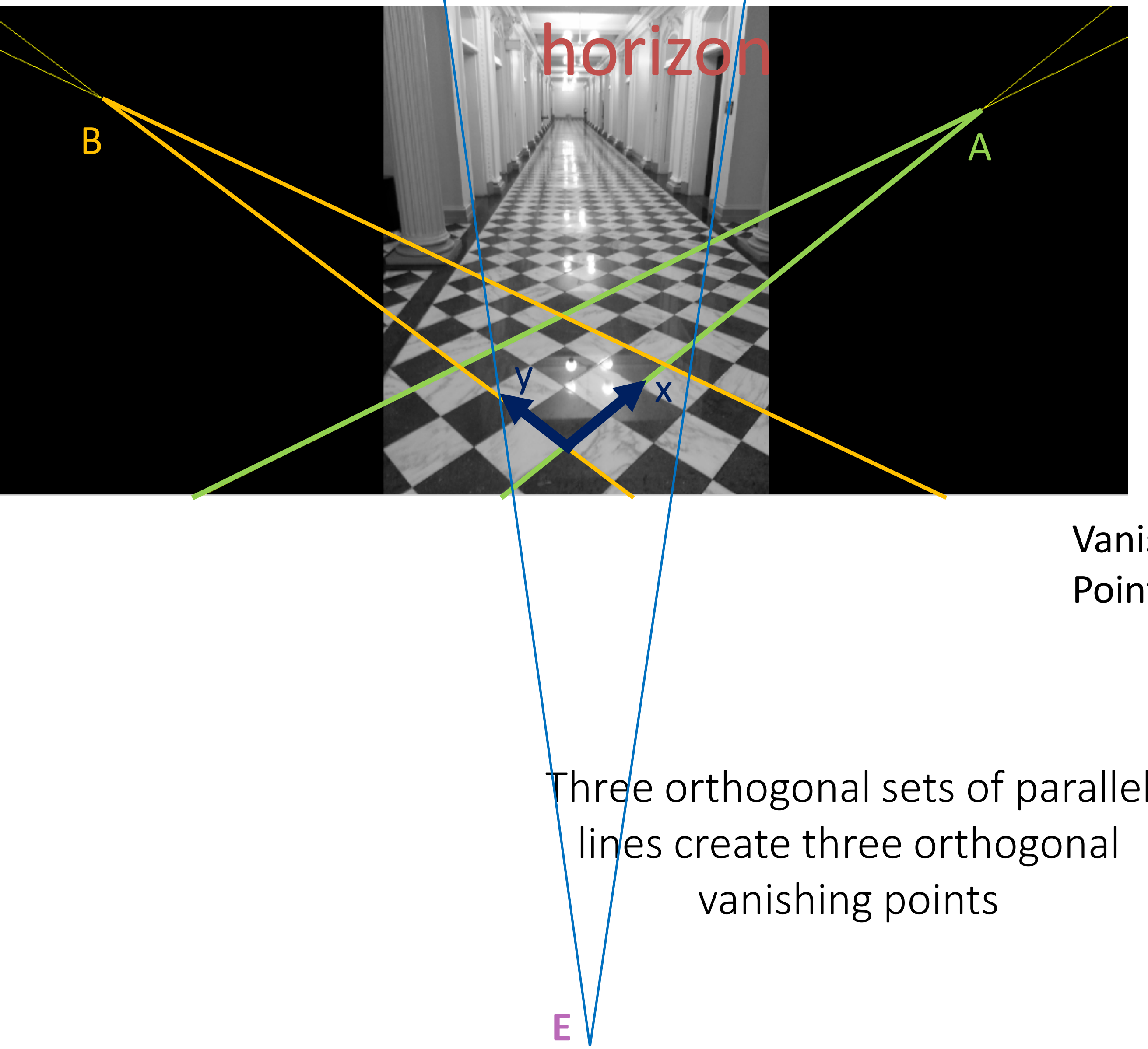
$$H \sim K(r_1, r_2, T)$$

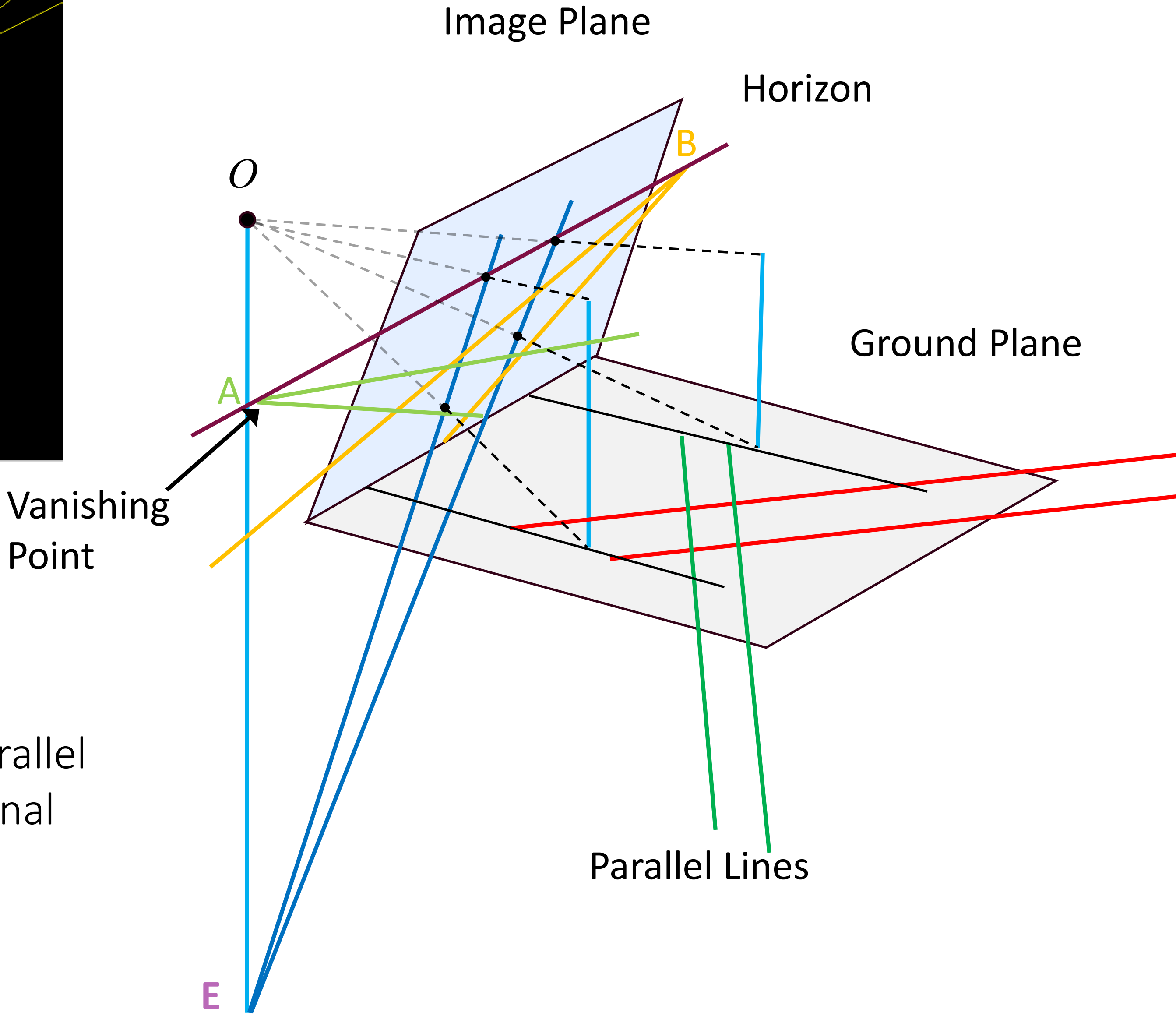
$$H \sim (a, b, c)$$

$$a \sim K r_1,\ b \sim K r_2 \longrightarrow r_1 \sim K^{-1} a,\ r_2 \sim K^{-1} b$$

How to compute intrinsics K from vanishing points

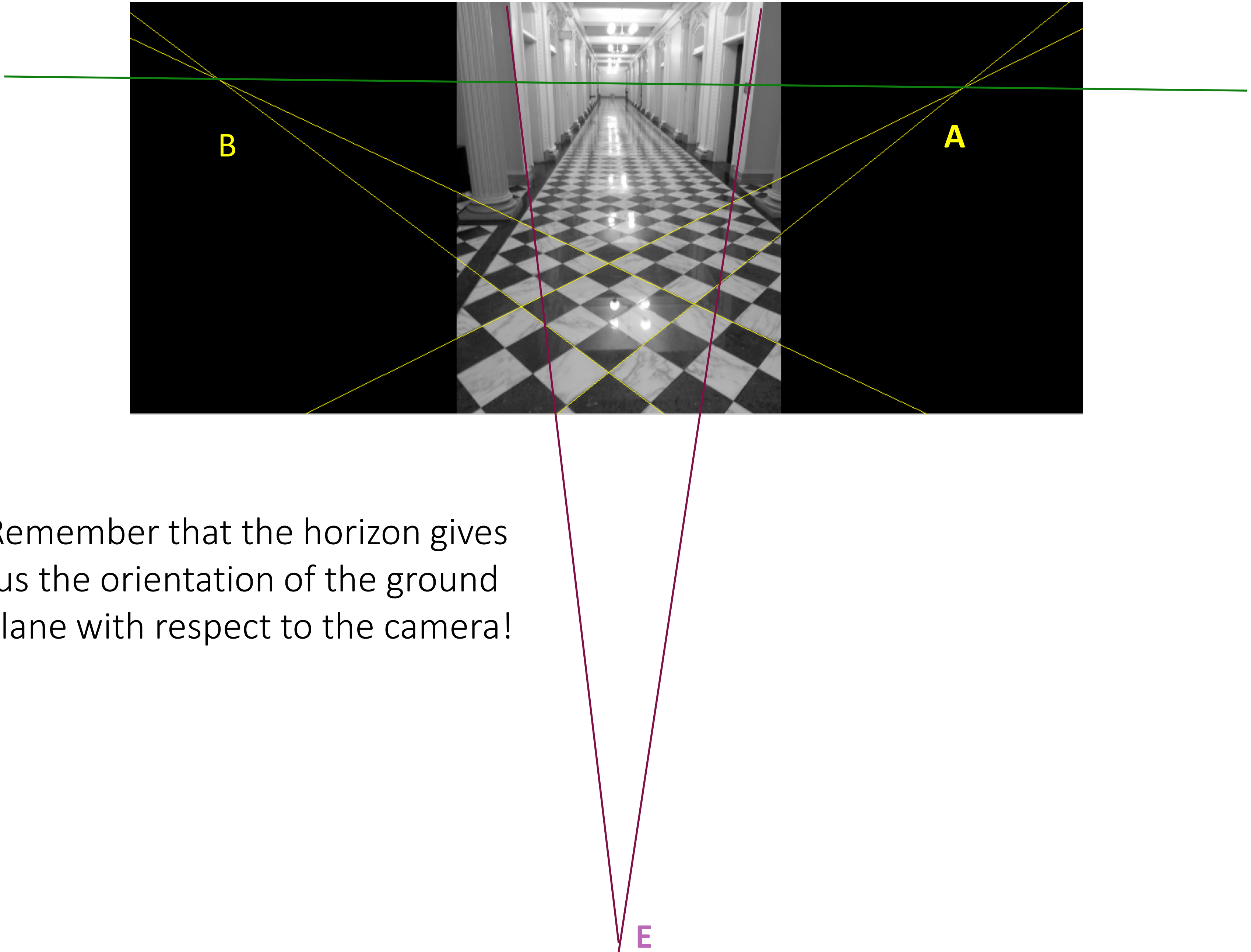
A scene with three orthogonal sets of parallel lines





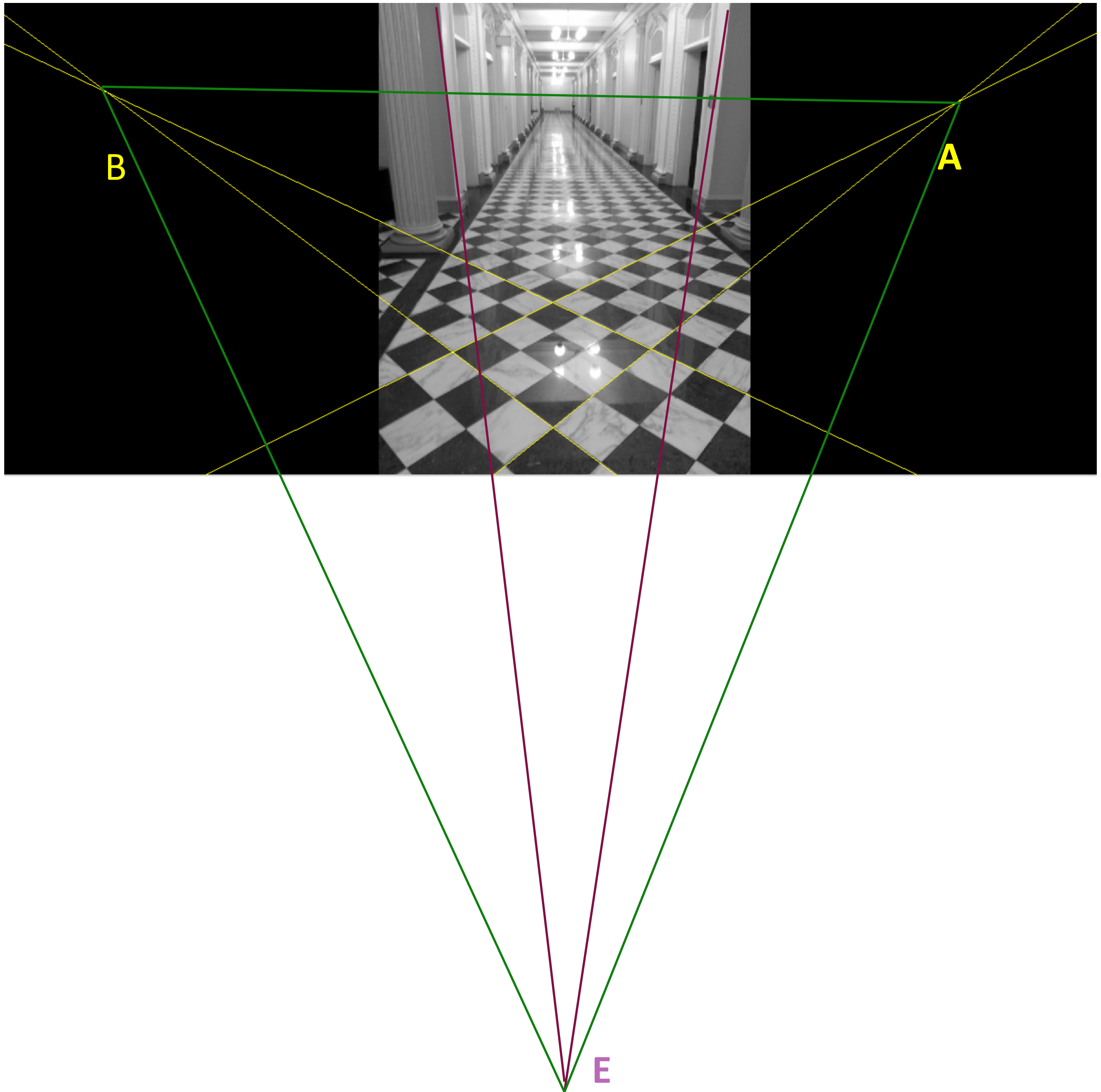
Three orthogonal sets of parallel lines create three orthogonal vanishing points

Line connecting AB is the horizon!

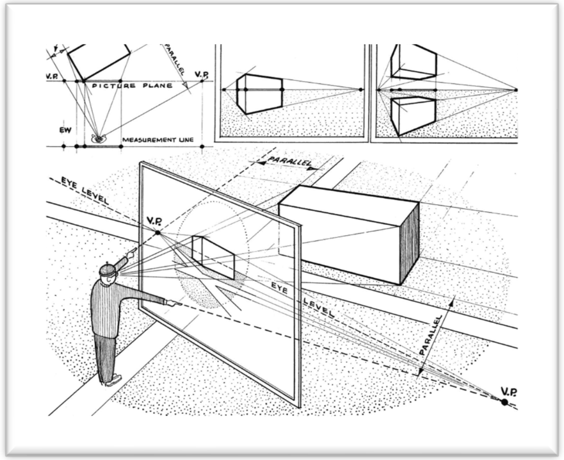


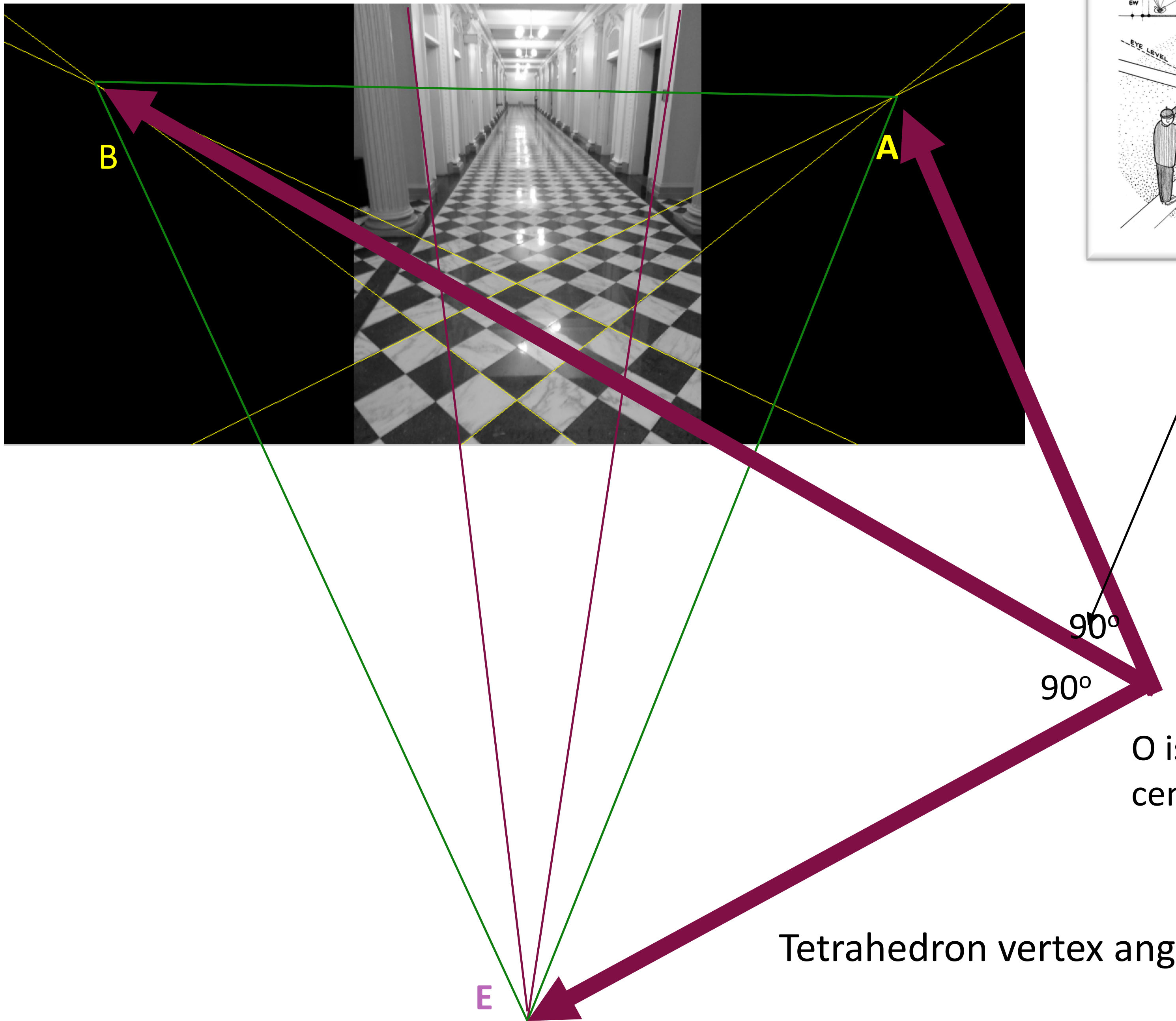
Remember that the horizon gives us the orientation of the ground plane with respect to the camera!

E is the vertical vanishing point!



Let's look at ABE as a tetrahedron OABE including the projection center

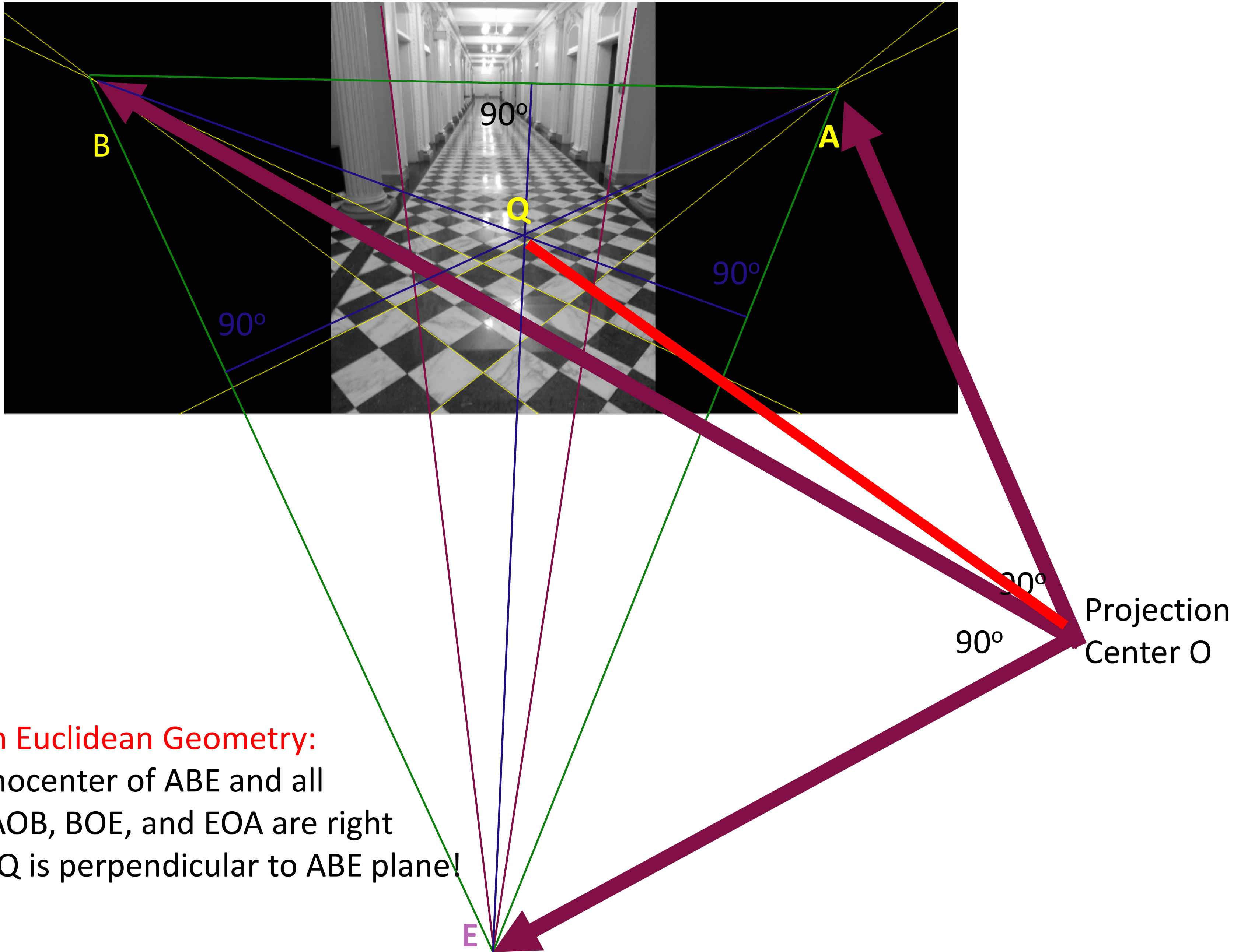




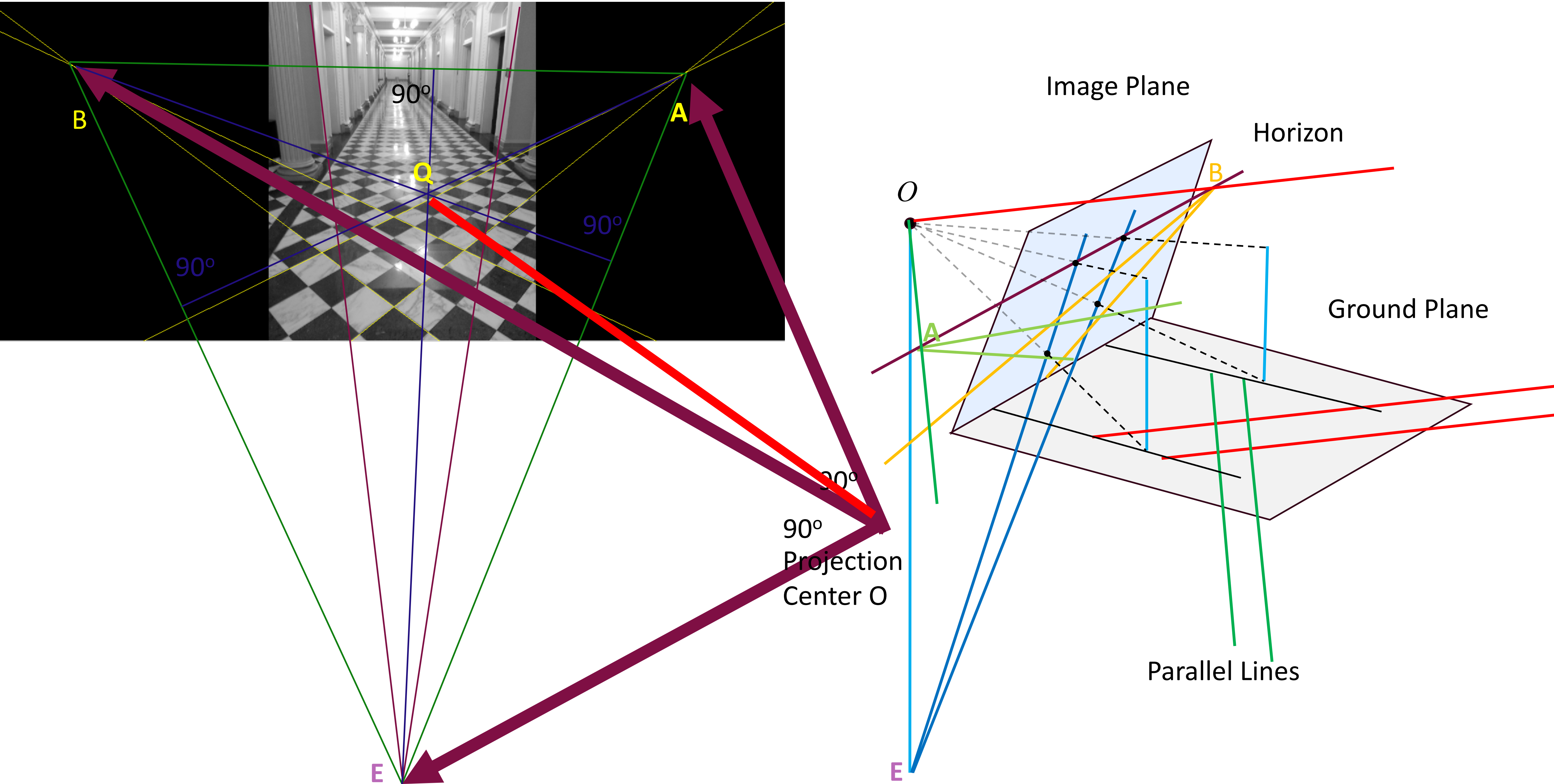
O is Projection center

Tetrahedron vertex angles are 90 degrees

Let Q be the orthocenter of the triangle ABE

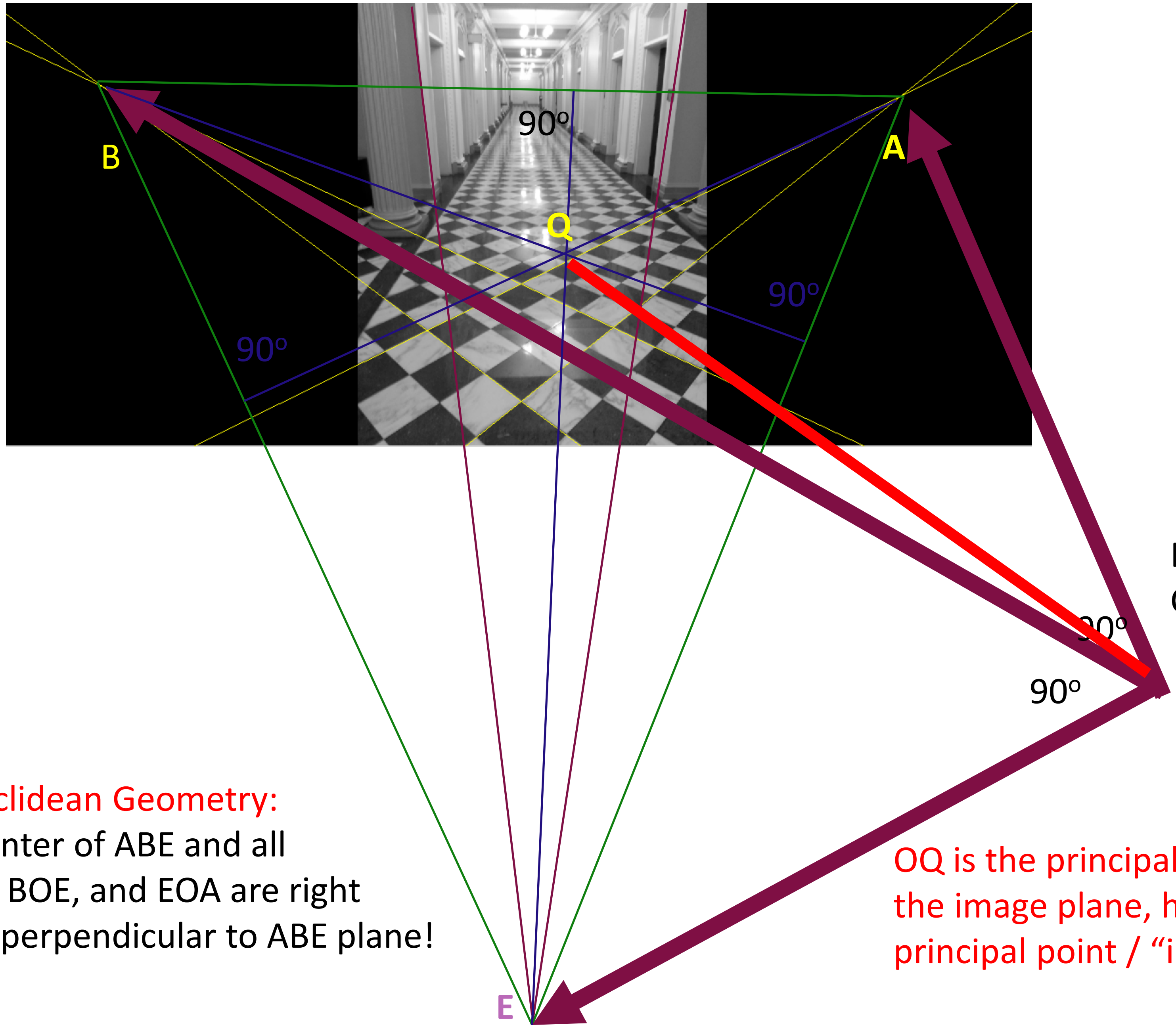


Theorem from Euclidean Geometry:
If Q is the orthocenter of ABE and all three angles AOB, BOE, and EOA are right angles , the OQ is perpendicular to ABE plane!



90º
B
A
Q
90º
90º
90º
90º
Projection
Center O
E
Image Plane
Horizon
B
O
A
Ground Plane
Parallel Lines
E

Let Q be the orthocenter of the triangle ABC



Theorem from Euclidean Geometry:

If Q is the orthocenter of ABE and all three angles AOB, BOE, and EOA are right angles , the OQ is perpendicular to ABE plane!

OQ is the principal axis and ABE is the image plane, hence, Q is the principal point / "image center"

Bonus proof: Perpendicular from camera center is the orthocenter of VPs

THEOREM: The image center (u_0, v_0) is the orthocenter of the triangle formed by the projections of three orthogonal vanishing points.

PROOF: See figure 2. Let $C = (u_0, v_0, 1)$ denote the homogeneous coordinates of the image center: it is defined as the intersection of image plane $V_1V_2V_3$ with the optical axis, which is the line through O and perpendicular to $V_1V_2V_3$.

$$OC \perp V_1V_2V_3 \Rightarrow OCV_1 \perp V_1V_2V_3$$
$$\Rightarrow OCV_1 \perp \text{any line contained in } V_1V_2V_3$$

By B: every plane containing OC is ⊥ to V1V2V3

By C: V2V3 ⊥ CV1, so OCV1 ⊥ V2V3

In particular $\boxed{OCV_1 \perp V_2V_3}$. Moreover, $OV_1 \perp OV_2$, therefore $\boxed{OCV_1 \perp OV_2V_3}$.

When two planes are perpendicular, their intersections with a third plane are also perpendicular. Therefore, the intersection of OCV_1 with $V_1V_2V_3$ is perpendicular to intersection of OV_2V_3 with $V_1V_2V_3$.

In other words $V_1C \perp V_2V_3$. A similar reasoning leads to $V_2C \perp V_3V_1$ and $V_3C \perp V_1V_2$, therefore C is the orthocenter of $V_1V_2V_3$.

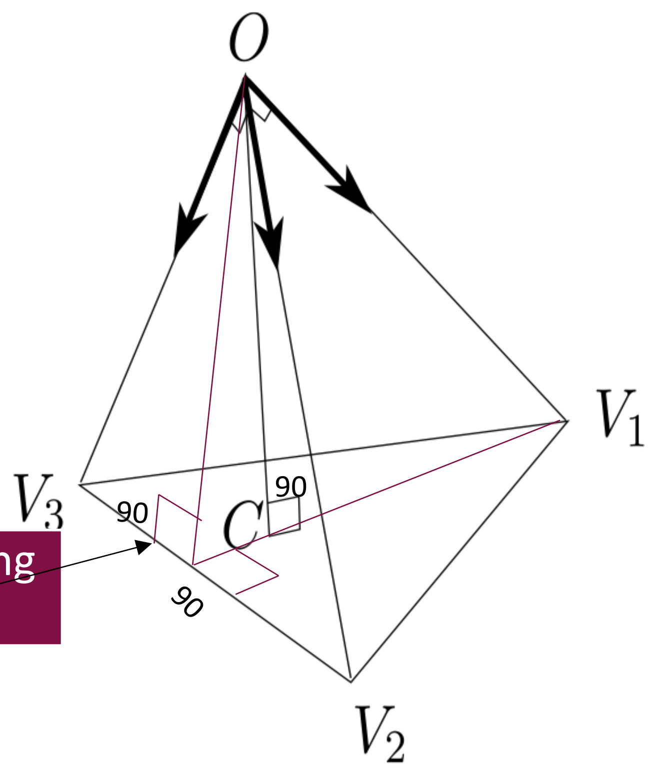


Figure 2: Three orthogonal vanishing points and image center.

We used the fact that the two non-parallel lines OV_2 and V_2V_3 contained in the plane OV_2V_3 are perpendicular to OCV_1, therefore $OV_2V_3 \perp OCV_1$.

We found the image center! What about the focal length (f=OQ)? Can it be computed from A,B, and E ?

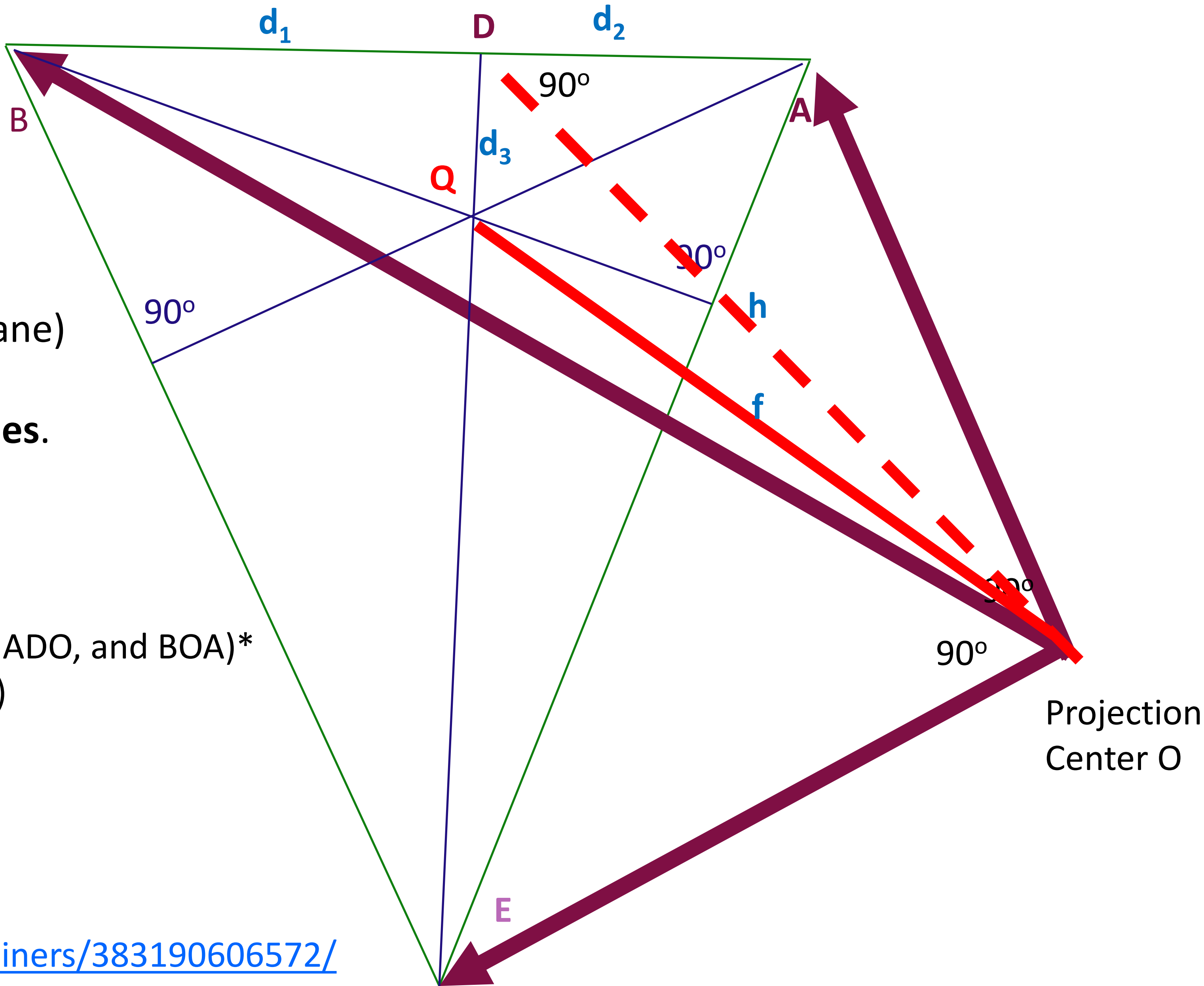
OQ ⊥ image plane.

So plane **OQD ⊥ image plane**

AD ⊥ ED (int. of OHD & img plane)

So, AD ⊥ OD

So, **BDO, ADO are right triangles**.



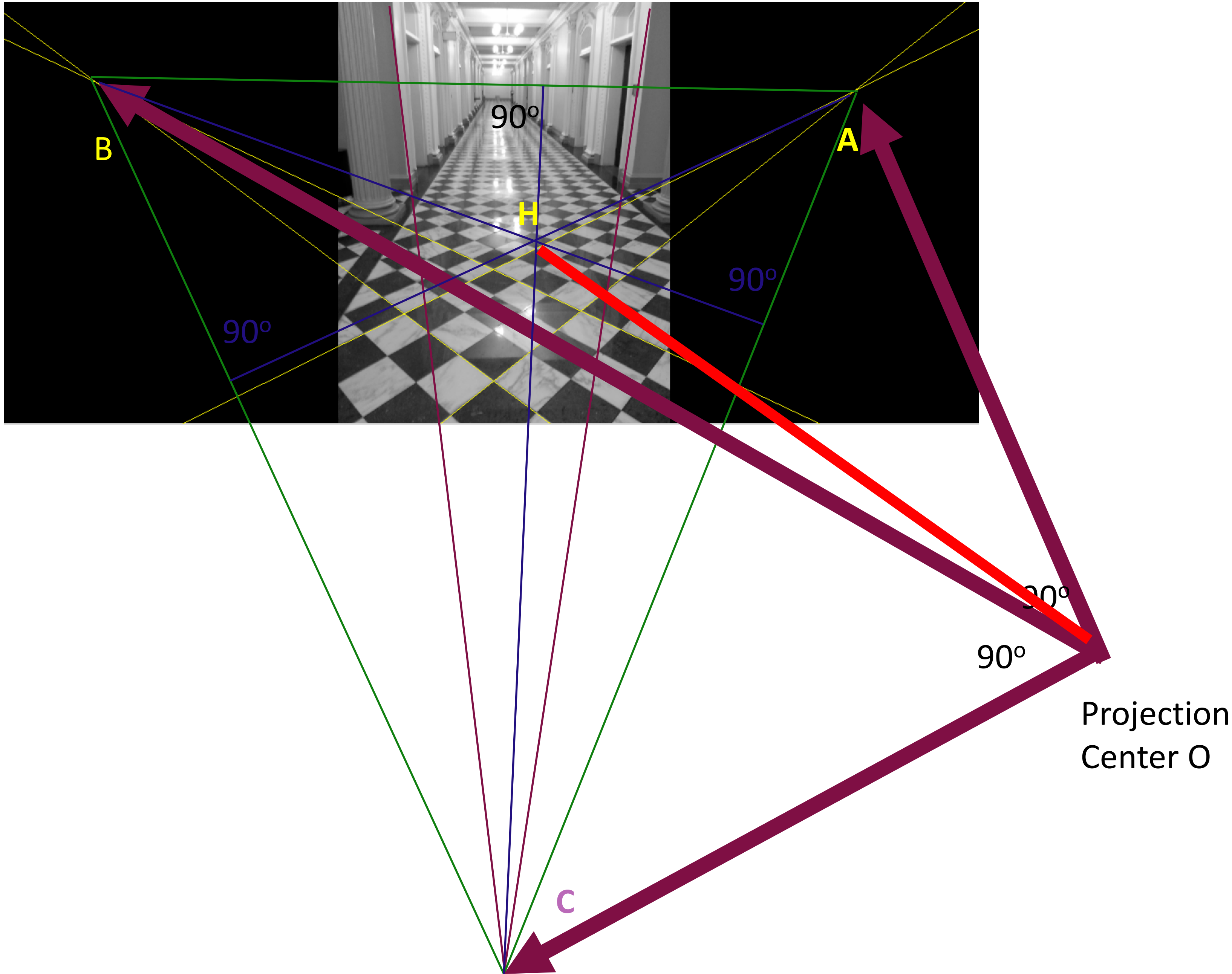
$h^2 = d_1 d_2$ (apply Pythagoras to BDO, ADO, and BOA)*

$f^2 + d_3^2 = h^2$ (Pythagoras again to OHD)

Hence $f^2 = d_1 d_2 - d_3^2$

* https://www.nagwa.com/en/explainers/383190606572/

Three orthogonal vanishing points (if none is at infty) allow computation of full intrinsics K (focal length and image center)!



Summary

- If we have 3 orthogonal VPs, we can get full intrinsics K (focal length and image center), and also extrinsics R.
  - What's missing? Just the translation t.
    - And that information is not contained in VPs, because camera translations don't affect the VPs!
    - Which is why, when we found homographies (that do contain full information about translation), we used more than just VPs.