CIS 5800

Machine Perception

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Lec 9: Feb 24, 2025

Recap: Projective Transformation = Homography = Collineation=Projectivity

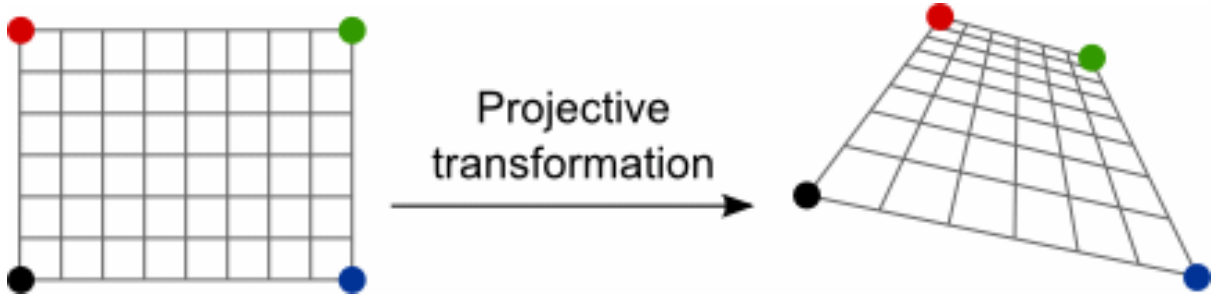
Definition

A **projective transformation** is any invertible matrix transformation $\mathbb{P}^2 \to \mathbb{P}^2$.

A projective transformation H maps p to $p' \sim Hp$

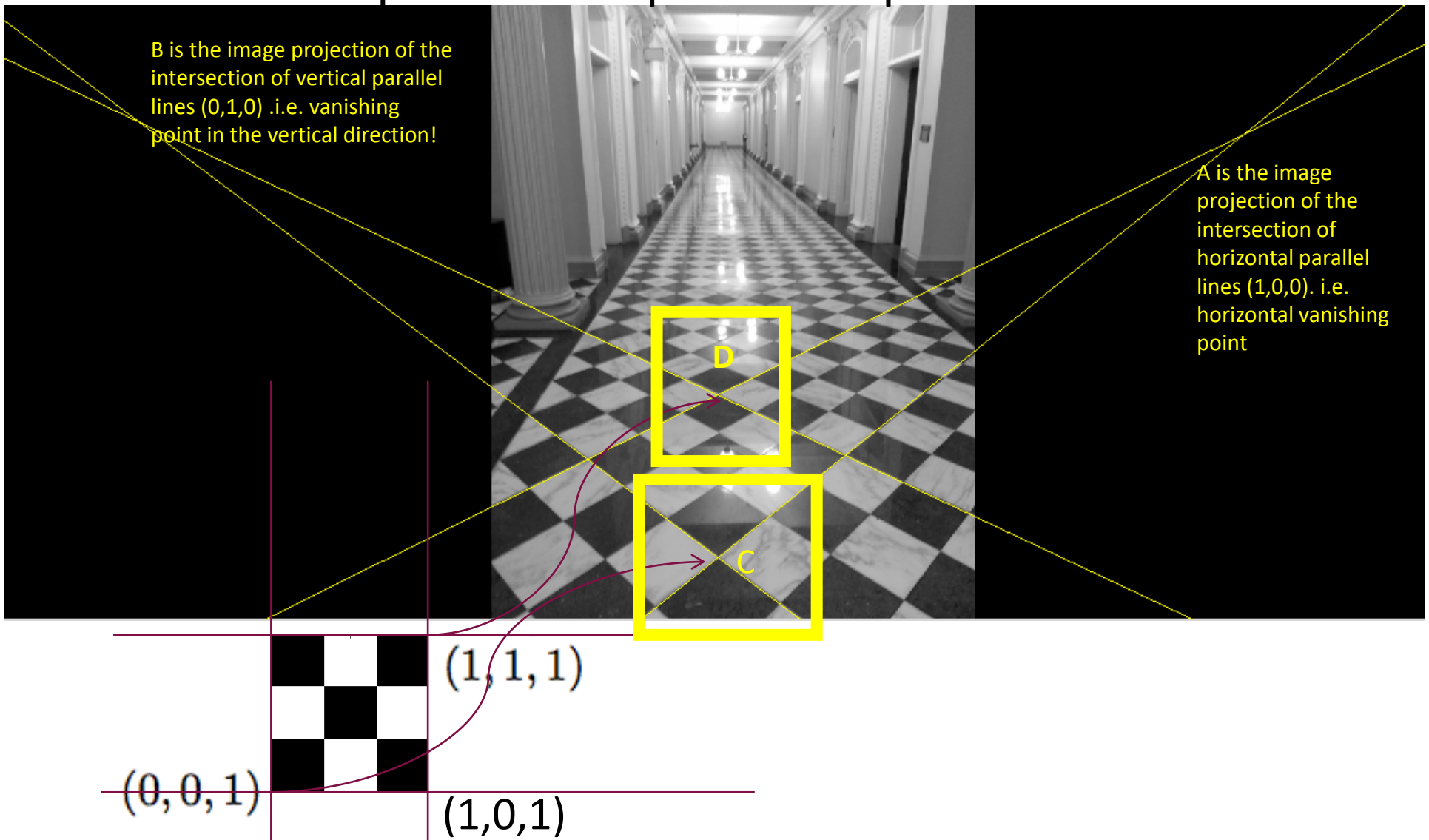
Invertibility means that $\det(H) \neq 0$ and that there exists $\lambda \neq 0$ such that $\lambda p' = Hp$

Observe that we will write either $p' \sim Hp$ or $\lambda p' = Hp$

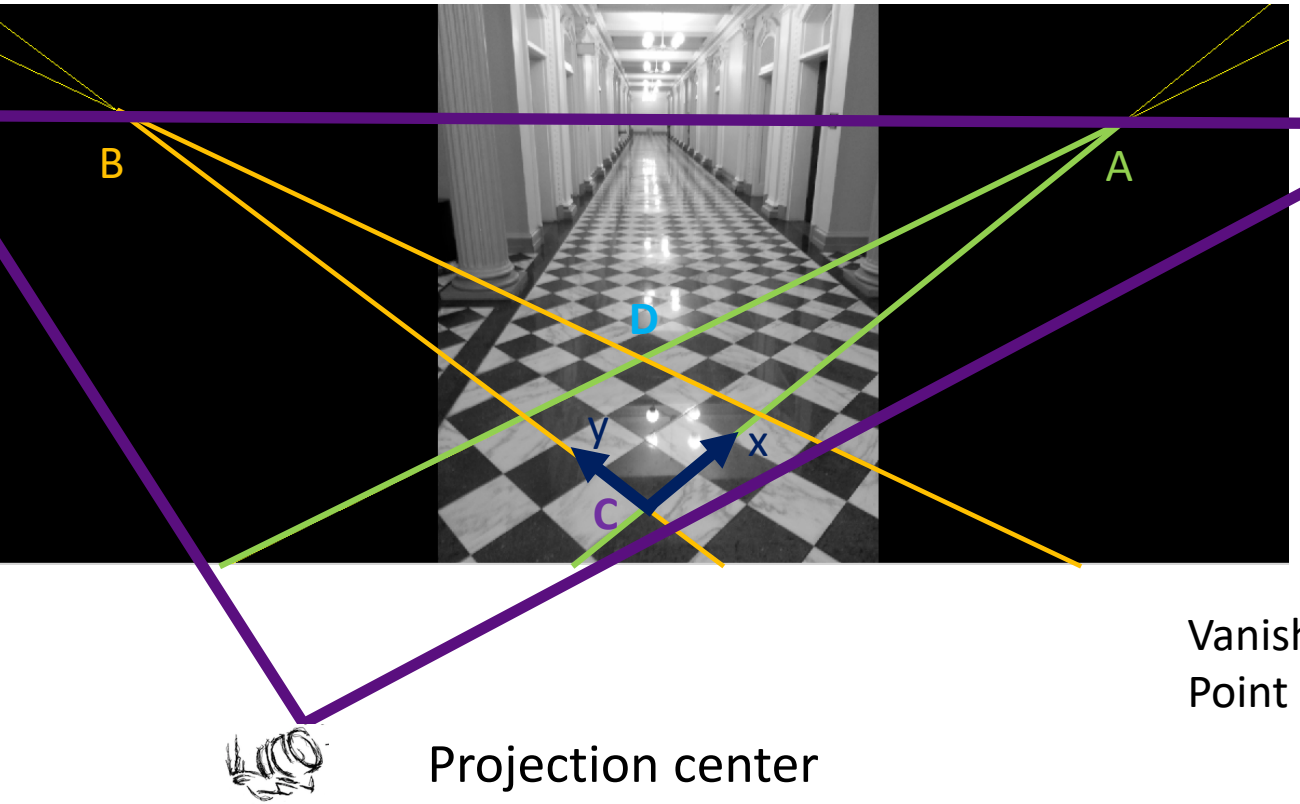


Recap: Computing Homography from 4 Point Correspondences

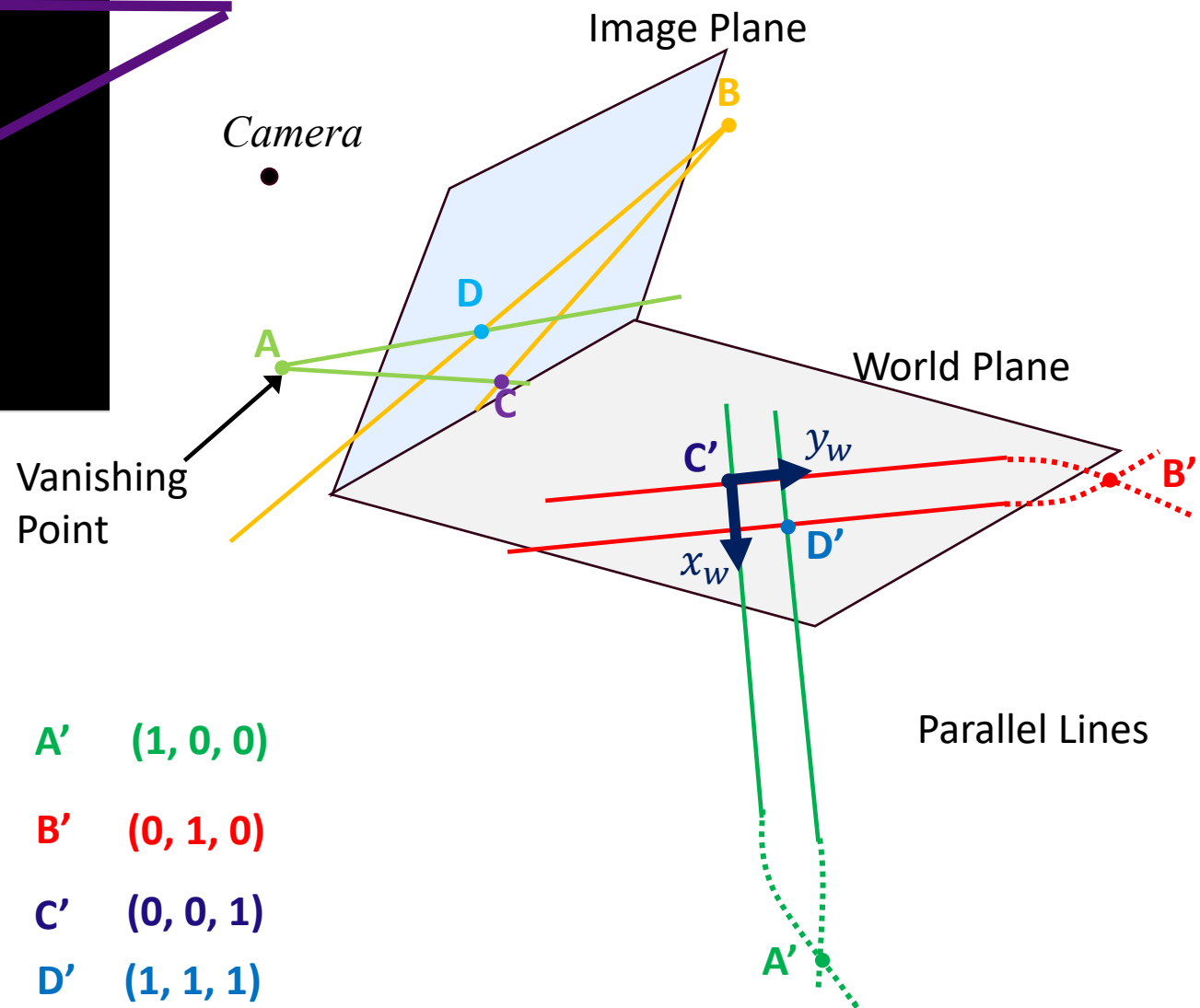
Note: makes sense, because after all, H has 8 degrees of freedom, and each 2D point correspondence pins down 2DOF.



Recap: Computing Homography from 4 Point Correspondences



Projection center



Correspondences:
($P^2 \rightarrow P^2$)

A	a	⟷	A'	(1, 0, 0)
B	b	⟷	B'	(0, 1, 0)
C	c	⟷	C'	(0, 0, 1)
D	d	⟷	D'	(1, 1, 1)

Recall: homography gives pose (given intrinsics K)

Pose from Projective Transformation

Recall the projection from world to camera

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim K \begin{pmatrix} r_1 & r_2 & r_3 & T \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}$$

and assume that all points in the world lie in the ground plane $Z = 0$.

Then the transformation reads

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim K \begin{pmatrix} r_1 & r_2 & T \end{pmatrix} \begin{pmatrix} X \\ Y \\ W \end{pmatrix}$$

And $r_3 = r_1 \times r_2$

Recap: But actually, not quite!

- According to the previous slide $K(r_1\ r_2\ T) = H$, or in other words,

$$K^{-1}H = (r_1, r_2, T) \text{ and } r_3 = r_1 \times r_2$$

If only life were so simple!

- Problem: when we ***estimate*** homographies (e.g. through solving linear systems with 2n equations from $n >= 4$ point correspondences), and then compute $K^{-1}H$, we aren't guaranteed to find a *valid* r_1 and r_2 pair. i.e. an orthonormal pair.
  - So, we need to find a way to first "correct" $(K^{-1}H)_{3\times3}$ to get orthonormal r_1 and r_2. Often called the "Procrustes", or "special orthogonal (SO) Procrustes" problem.
  - And we must solve this in real-time for robotics applications, so preferably an inexpensive approach.

Recap: Full Kabsch algorithm for finding pose via homography

1. Find H up to a scale factor from the point coorrespondences

2. Compute $H' = K^{-1}H$. Let H''s columns be $\begin{pmatrix} a & b & c \end{pmatrix}$

3. Minimize

$$\| \begin{pmatrix} a & b & c \end{pmatrix} - \lambda \begin{pmatrix} r_1 & r_2 & T) \end{pmatrix} \|_F$$

w.r.t. $\lambda \in \mathbb{R}, r_1, r_2, T \in \mathbb{R}^3$
s.t. $r_1^T r_2 = 0$ and $\|r_1\| = \|r_2\| = 1$

Let

$$(a\ b) = U_{3x2} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} V_{2x2}^T.$$

Then

$$\begin{pmatrix} r_1 & r_2 \end{pmatrix} = U_{3x2} V_{2x2}^T \qquad \text{and} \qquad \lambda = \frac{s_1 + s_2}{2}$$

4. $T = c/\lambda$ and $R = \begin{pmatrix} r_1 & r_2 & r_1 \times r_2 \end{pmatrix}$. Scale R to have determinant 1 if needed.

Pose from Point Correspondences, the Perspective N Point Problem (PnP)

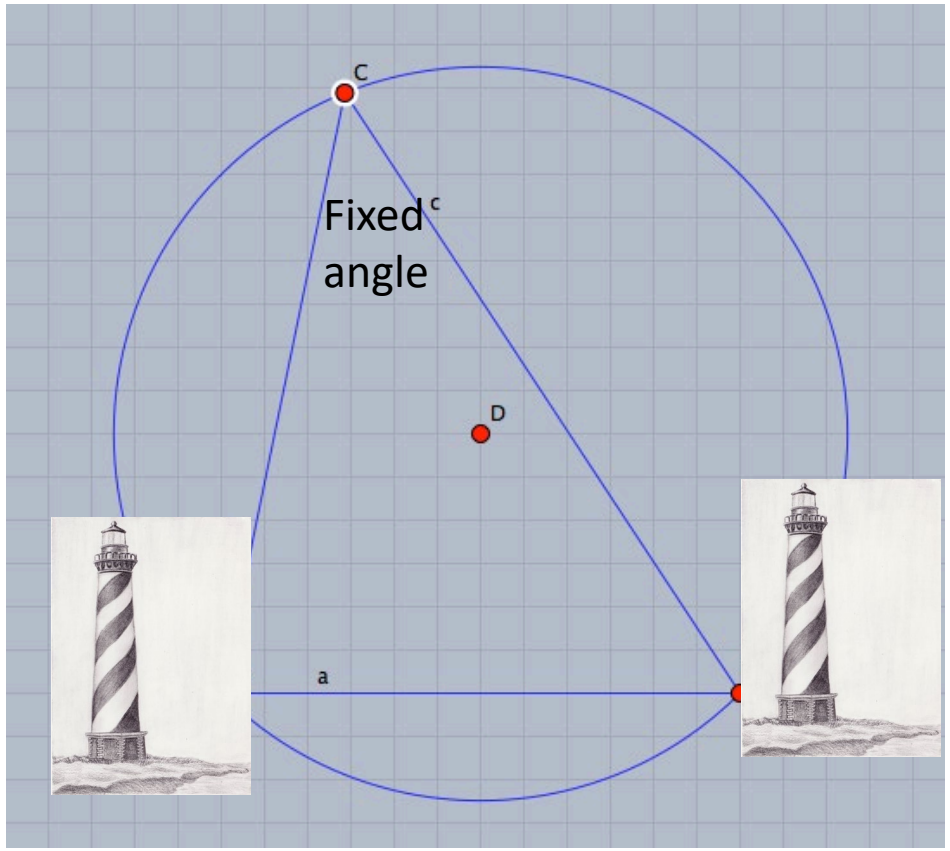
Localization by observing known 3D points from the world?

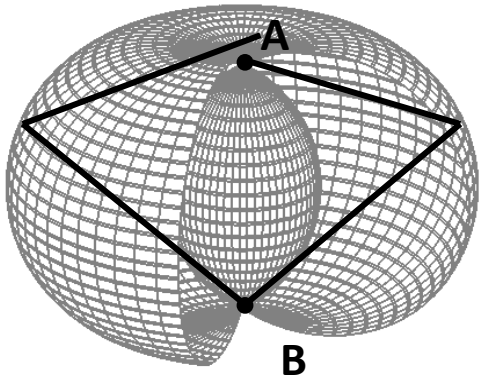
A real problem for autonomous cars, for example!
GPS: ~ a few feet accuracy. Just not good enough.

Instead, autonomous cars rely on 3D maps of the world to localize!

Navigation with “bearings” from 2 points

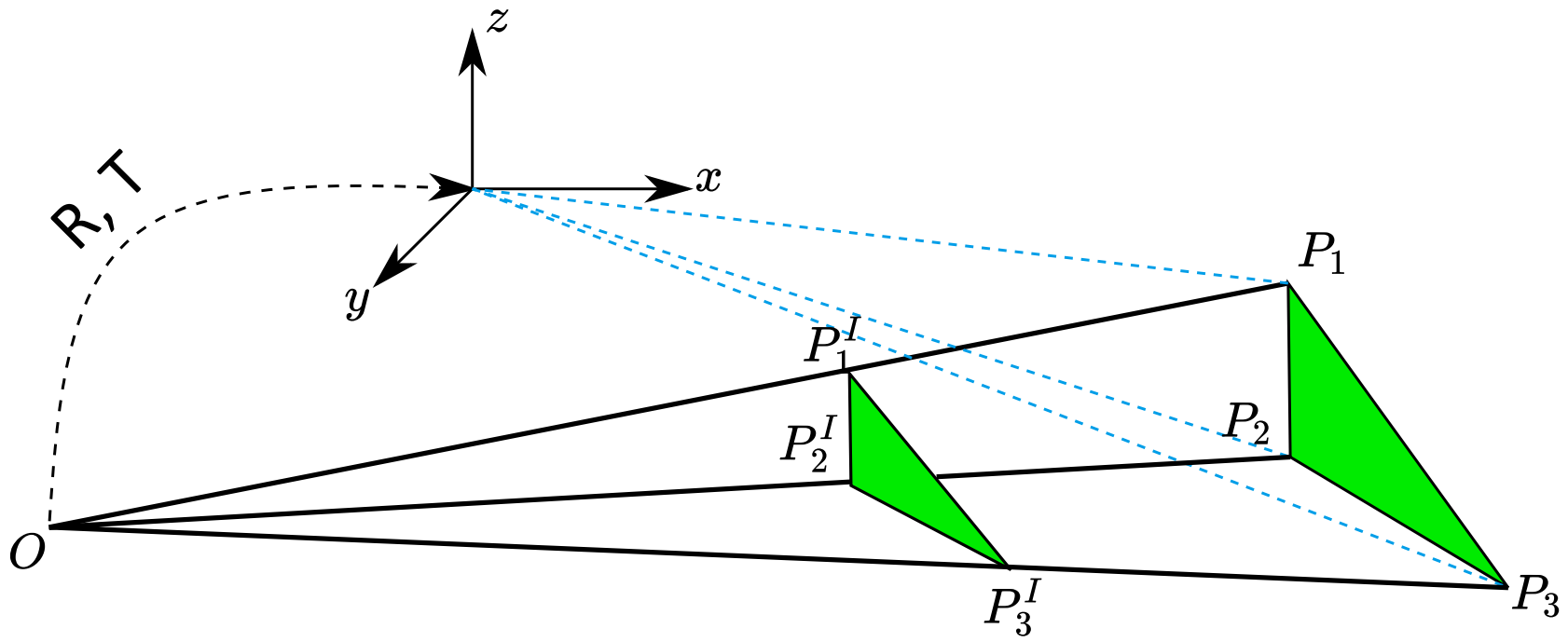
If I observe two lighthouses being some fixed angle θ apart, where am I?





Two points are not sufficient.
Camera could be anywhere on an upper semicircle in 2D.
Worse in 3D: anywhere on a toroid in 3D. Need more points!

The *Perspective* 3-Point Problem



- Given the point correspondences, find camera pose R, T

What are the differences from 4-Point Algorithm?

Jingnan Shi, https://jingnanshi.com/blog/pnp_minimal.html

P3P v.s. Homography

- Why P3P needs only three point correspondences, while computing Homography needs four point correspondences?

Simplified 3-Point Problem w. 3D Camera Coordinates

A triangle's world 3D coordinates $P_i \in \mathbb{R}^3$ are known, and its camera-centric 3D coordinates $P_i^c \in \mathbb{R}^3$ are known

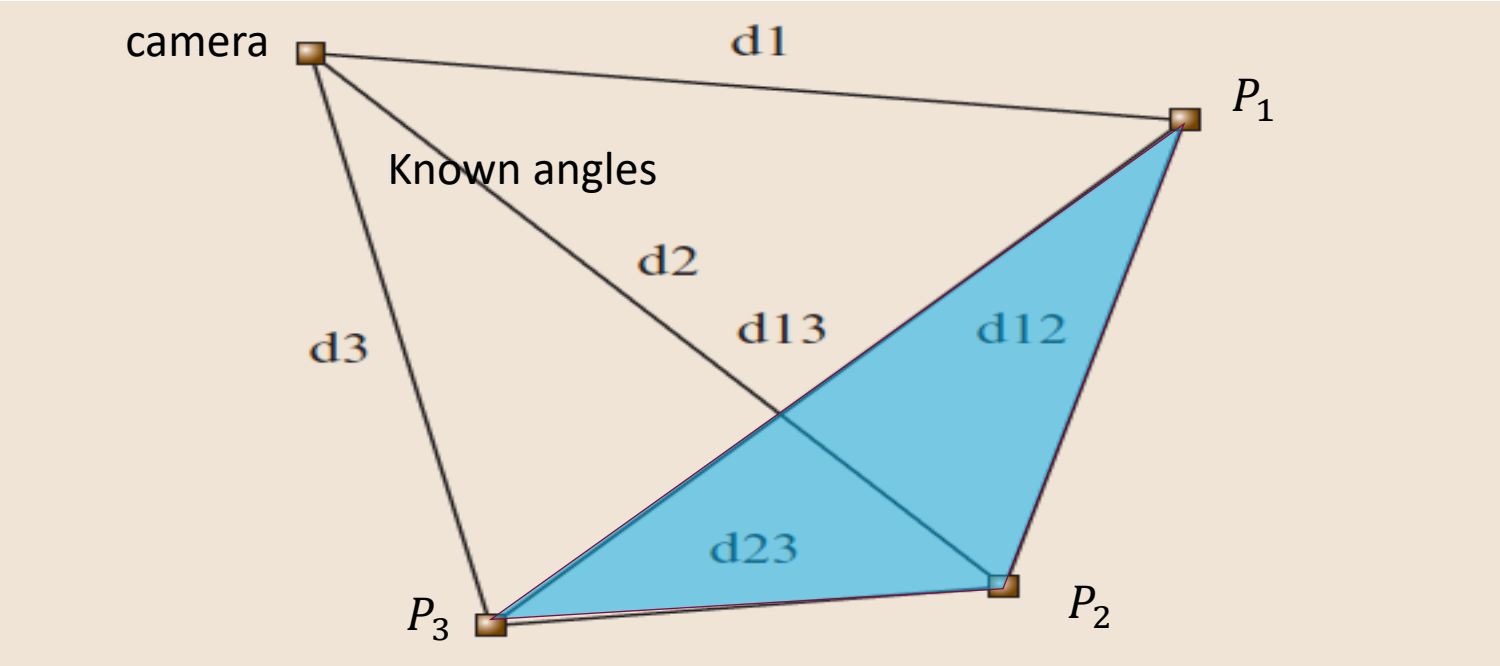
The 3D->3D 3-Point Problem: Find camera pose R, T such that

$$P_1^c = RP_1 + T$$
$$P_2^c = RP_2 + T$$
$$P_3^c = RP_3 + T$$

Full camera coordinates may come from depth cameras, but otherwise, we only have pixel coordinates.

Plan: starting from only pixel coordinates, first reduce the problem to 3D->3D.

P3P from Pixels



A triangle's world coordinates P_i are known, and its pixel coordinates are known

Pixels → "Calibrated coordinates"

RGB images only provide pixel coordinates u, v for each vertex, not camera-centric 3D coordinates. Convert these to:

"Calibrated coordinates": $\boldsymbol{p_i} \sim K^{-1}(u_i \quad v_i \quad 1)^T$

These are essentially image plane coordinates with principal point as origin, and focal length set to 1.

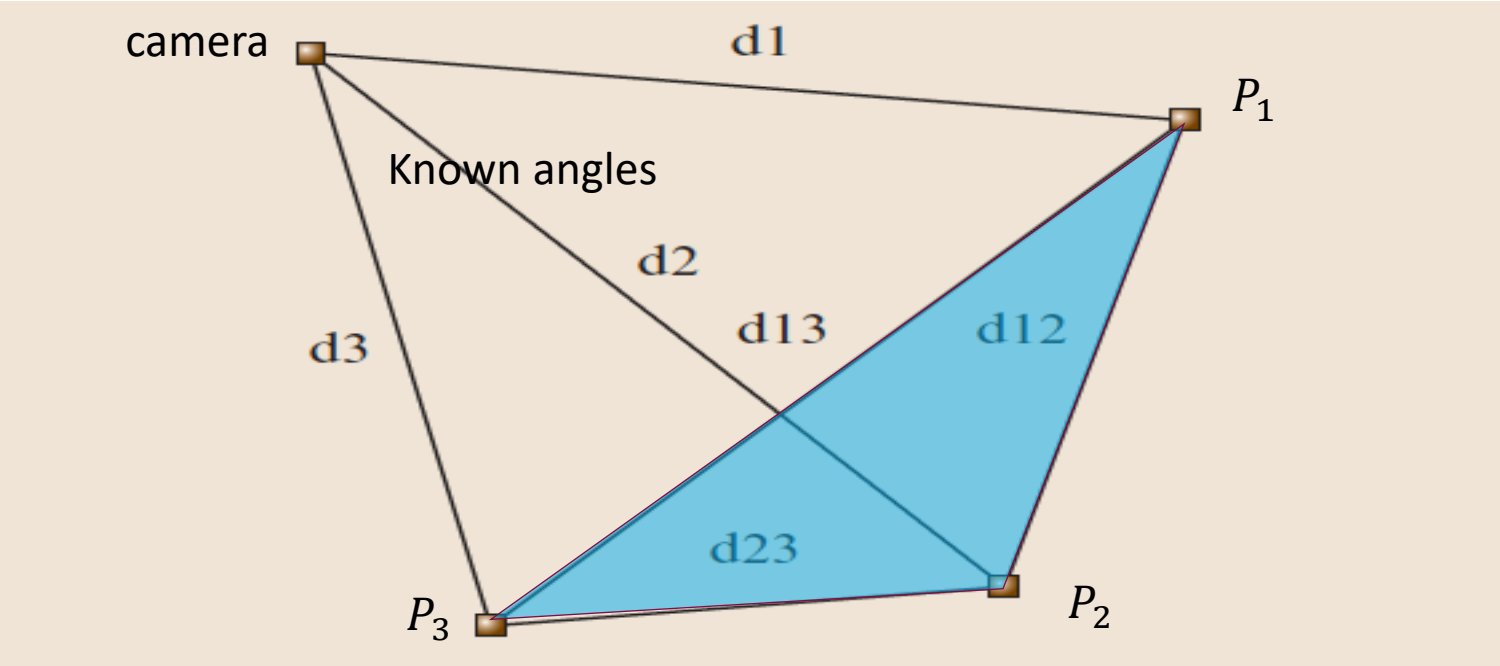
Euclidean interpretation: $\boldsymbol{p_i}$ is a vector in camera coordinates, originating from the camera center and pointing toward the 3D point corresponding to pixel coordinates (u_i, v_i).

Calibrated coordinates + distance = Camera-centric 3D

- After writing as calibrated coordinates, camera-centric 3D coordinates is just one step away: finding the distance of the 3D point along that direction, i.e. “depth”

- To reduce P3P to 3D->3D as we had planned, we will need to find those depths. Indeed, that is the first step of P3P.

P3P from ~~Pixels~~ Calibrated Coordinates



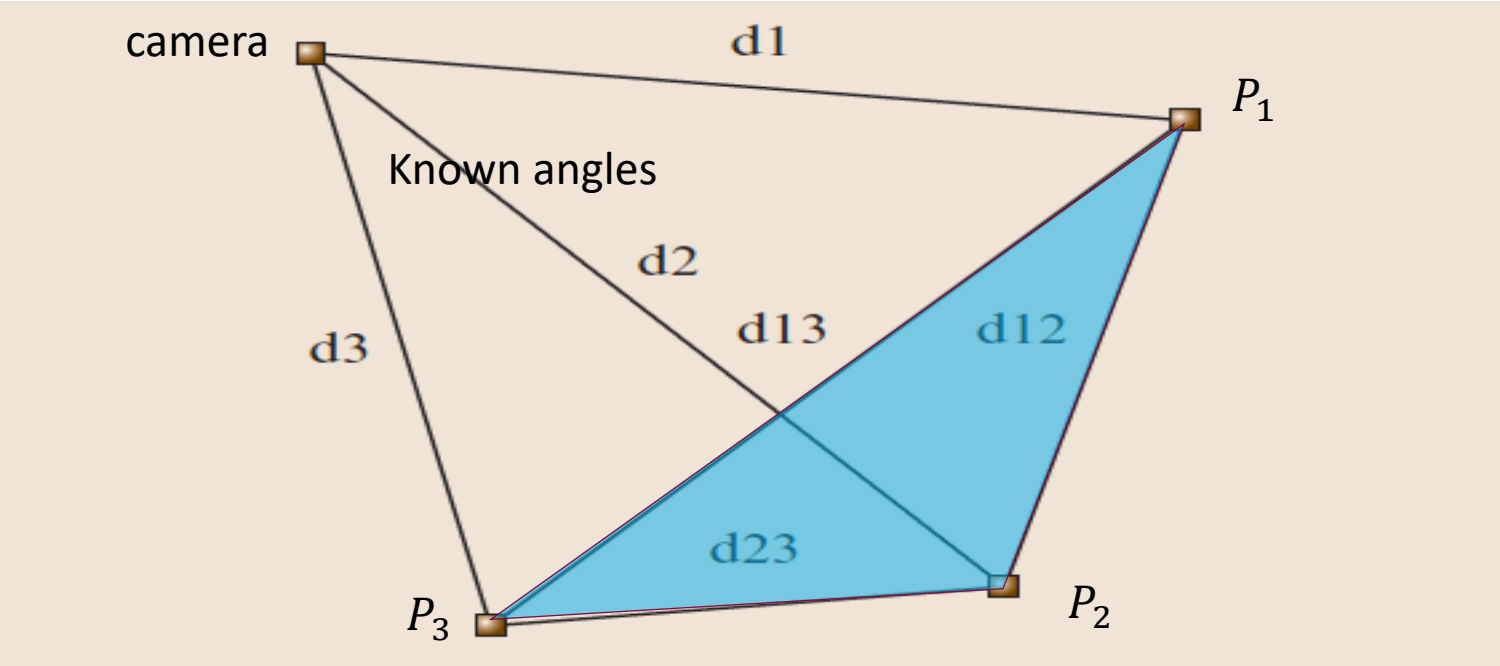
A triangle's world coordinates P_i are known, and its camera-centric calibrated coordinates p_i are known

The P3P problem: Find λ_i, R, T such that

$$\lambda_1 p_1 = RP_1 + T$$
$$\lambda_2 p_2 = RP_2 + T$$
$$\lambda_3 p_3 = RP_3 + T$$

Q: Are λ_i the same as "depths" d_i?
A: No, because p_i are not unit vectors.

P3P from Calibrated Coordinates



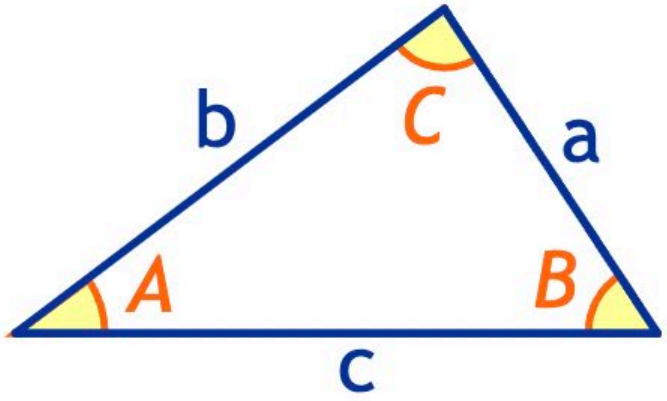
A triangle's world coordinates P_i are known, and its camera-centric calibrated coordinates p_i are known

The P3P problem: Find d_i, R, T such that

$$\frac{d_i}{||p_i||_2} p_i = RP_i + T, \qquad \forall i = 1,2,3$$

High School Flashback: Triangle Cosine Law

For any triangle:



a, **b** and **c** are sides.

C is the angle opposite side c

The Law of Cosines (also called the **Cosine Rule**) says:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

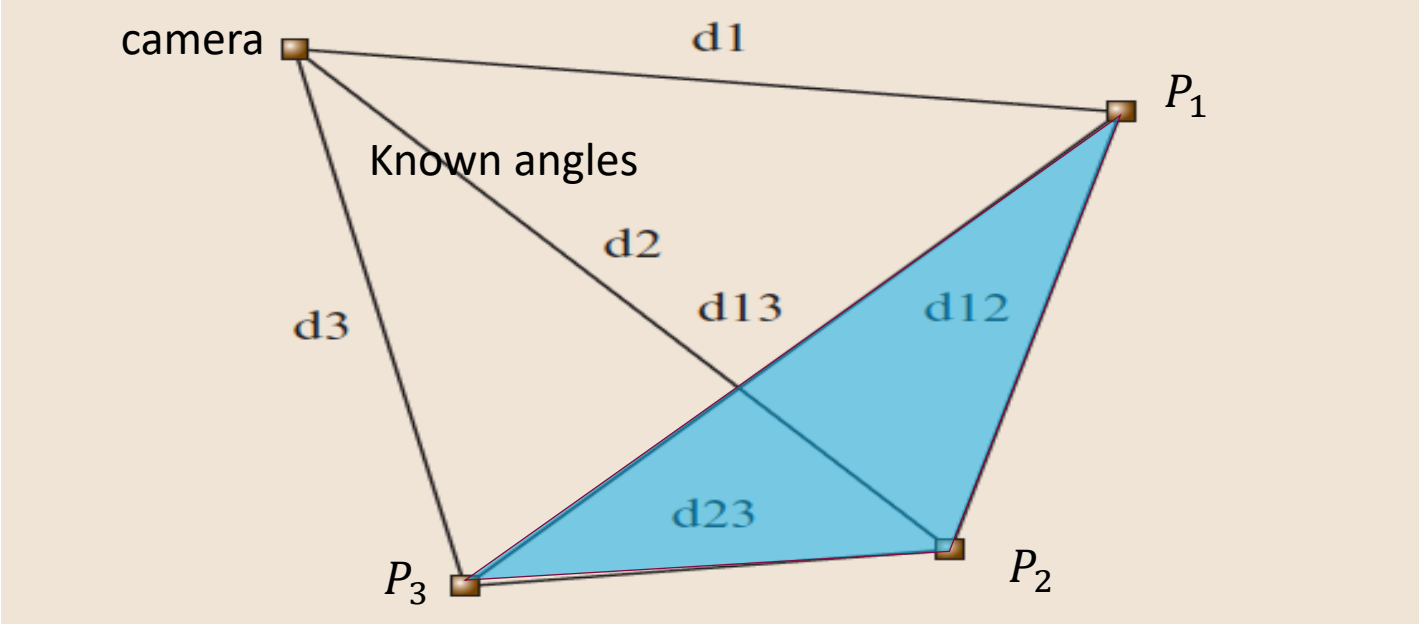
https://www.mathsisfun.com/algebra/trig-cosine-law.html

P3P Step 1: Finding depths d_i of triangle vertices

Let δ_{ij} denote the observed angle between the calibrated coordinates p_i and p_j

Then cosine law reads

$$d_i^2 + d_j^2 - 2d_i d_j \cos\delta_{ij} = d_{ij}^2$$



The cosine law

$$d_i^2 + d_j^2 - 2 d_i d_j \cos \delta_{ij} = d_{ij}^2$$

applies for each point pair. With 3 points we could solve 3 quadratic equations for $d_{i=1\ldots3}$.

Set $d_2 = u d_1$ and $d_3 = v d_1$ and solve all three equations for d_1:

$$\begin{aligned} d_1^2 &= \frac{d_{23}^2}{u^2 + v^2 - 2uv \cos \delta_{23}} \\ d_1^2 &= \frac{d_{13}^2}{1 + v^2 - 2v \cos \delta_{13}} \\ d_1^2 &= \frac{d_{12}^2}{u^2 + 1 - 2u \cos \delta_{12}} \end{aligned}$$

P3P Step 1: The algebraic drudgery

Reduces to two quadratic equations in u and v.

$$d_{13}^2(u^2+v^2-2uv\cos\delta_{23}) = d_{23}^2(1+v^2-2v\cos\delta_{13}) \quad (1)$$

$$d_{12}^2(1+v^2-2v\cos\delta_{13}) = d_{13}^2(u^2+1-2u\cos\delta_{12}) \quad (2)$$

a) Solve Eqn (1) for u^2 in terms of u, v, v^2 (and constants).
b) Plug this solution into Eqn (2), so that it has no u^2 term. Solve for u in terms of v, v^2, and constants.
c) Plug this solution for u back into Eqn (1), so that it has no more u or u^2. Instead, it is a 4th degree equation in v. *Get the 4 real solutions analytically.*
d) Then plug back into the solution found in step b) above, to get u.
e) Then get d_1 from the quadratic equations on the last page.
f) Then plug back into $d_2 = ud_1$ and $d_3 = vd_1$ from the last page to get d_2, d_3.

Reference for full version of P3P algebraic drudgery

- Grunert 1841, summarized in Haralick et al, 1993 which is linked in supplementary readings.

Review and Analysis of Solutions of the Three Point Perspective Pose Estimation Problem

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End result of algebraic drudgery

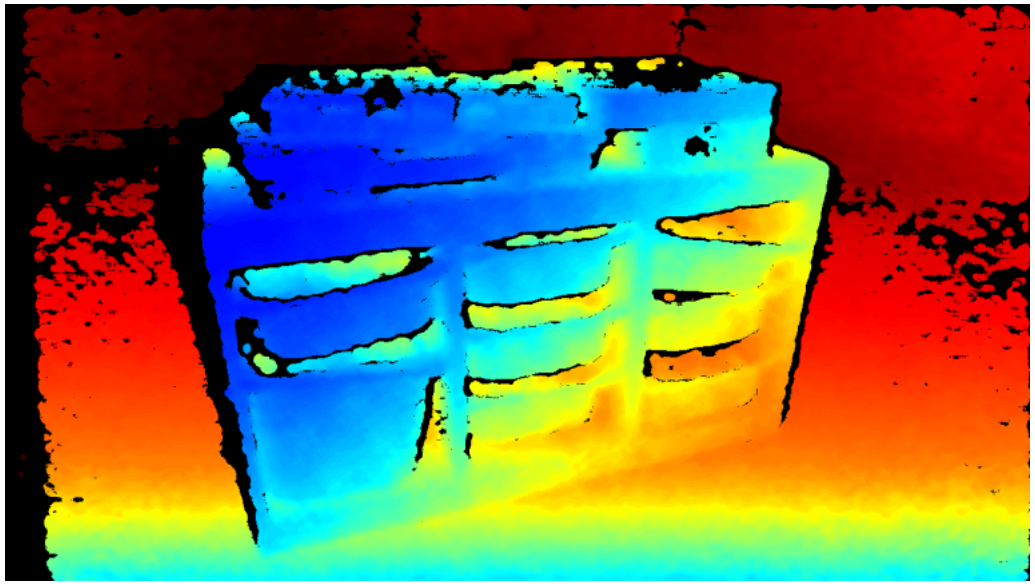
- We know the distances from camera to all three points! This means that we have effectively recovered “depth”, which combined with the calibrated coordinates we had before, means that we have:

Camera-centric 3D coordinates for triangle vertices!

So, we can now do what we planned all along, and just solve the 3D->3D 3-Point Problem.

Aside: Depth cameras

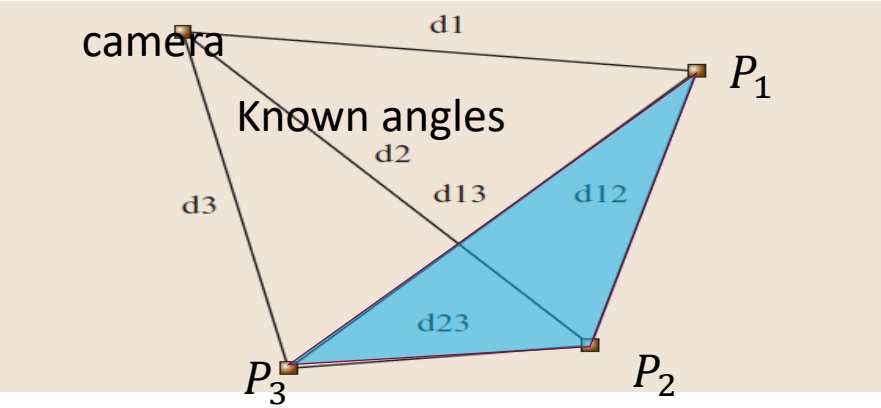
If we had started with depth images, we could have directly skipped to this point!



Note: Depth cameras commonly work from “stereo”: 2-view geometry!

Intel RealSense: https://www.intelrealsense.com/stereo-depth-vision-basics/

P3P Step 2: 3D->3D Pose/ 3D Registration. Find R&T!



The P3P problem has reduced to:

Find R, T such that

$$\frac{d_i}{||p_i||_2} p_i = RP_i + T, \qquad \forall i = 1,2,3$$

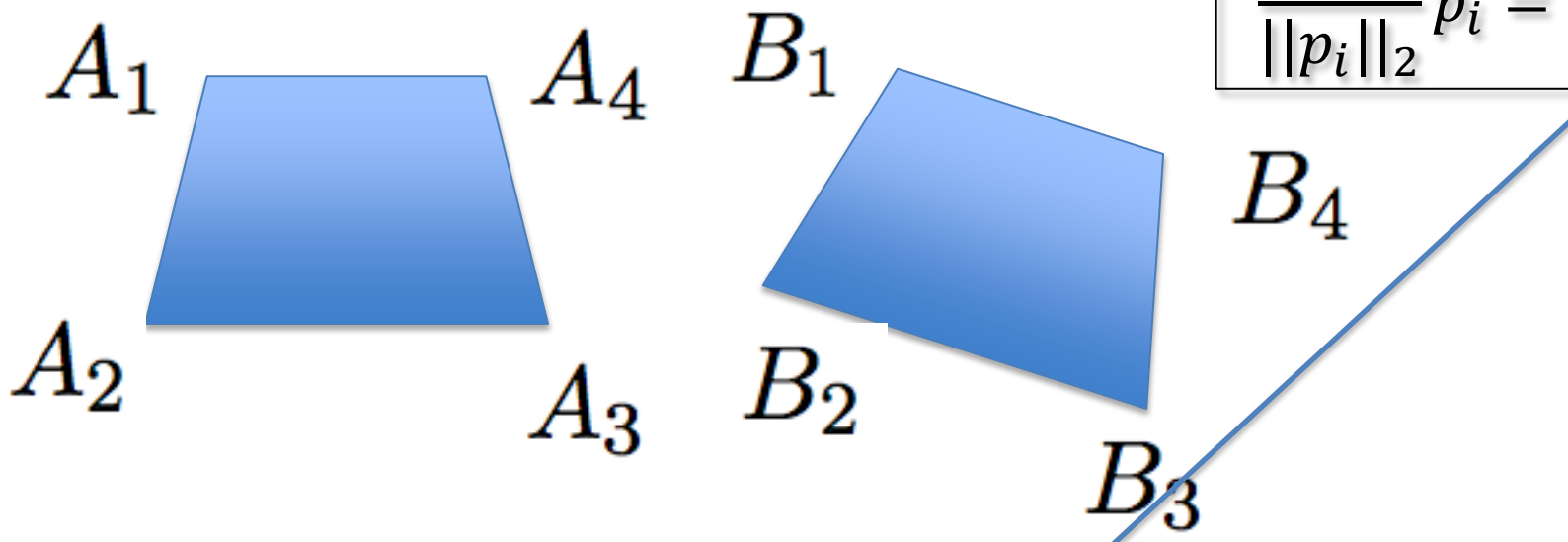
But naïve direct solution of the linear system is perilous, because rotation matrix R might not be valid.

(Does this remind you of something?)

Procrustes Problem and Its Kabsch Algorithm

The P3P problem: Find R, T such that

$$\frac{d_i}{||p_i||_2} p_i = RP_i + T, \qquad \forall i = 1,2,3$$



How do we solve for R, T from n point correspondences?

$$A_i = RB_i + T$$

(Note: the method we discuss for P3P step 2 is the same for >3 points too. So the rest of this discussion applies to more points too.)

(Nothing new on this slide except change of notation)

We solve a minimization problem for $N > 3$ point correspondences:

$$\min_{R,T} \sum_{i}^{N} \|A_i - RB_i - T\|^2$$

After differentiating with respect to T we observe that the translation is the difference between the centroids:

$$T = \frac{1}{N} \sum_{i}^{N} A_i - R \frac{1}{N} \sum_{i}^{N} B_i = \bar{A} - R\bar{B}$$

Notes:

1. If we find the right rotation matrix R, we will be able to find T afterwards.
2. If the centroids $\bar{A}$ and $\bar{B}$ had both been zero, then we could have set $T = 0$.

3. Rotation matrix between point sets is unaffected by translating the point sets, so can centroid-subtract the point sets before finding rotation.

4. And after this centroid-subtraction, we know $T = 0$ (by note#2 above), so:

$$\min_R \|A - RB\|_F^2$$

where

$$A = \begin{pmatrix} A_1 - \bar{A} & \dots & A_N - \bar{A} \end{pmatrix}$$

and

$$B = \begin{pmatrix} B_1 - \bar{B} & \dots & B_N - \bar{B} \end{pmatrix}$$

Reducing to Procrustes problem

This problem can be shown to be equivalent to the Procrustes problem we have seen last class for finding 2D homographies!

$$\underset{R \in SO(3)}{\operatorname{argmin}} ||A - RB||_F^2 = \underset{R \in SO(3)}{\operatorname{argmin}} ||R - AB^T||_F^2$$

Kabsch algorithm for Procrustes (proof in supp readings)

$$\underset{R \in SO(3)}{\arg\min} \|R - \begin{pmatrix} h_1' & h_2' & h_1' \times h_2' \end{pmatrix}\|_F^2$$

If the SVD of

$$\begin{pmatrix} h_1' & h_2' & h_1' \times h_2' \end{pmatrix} = USV^T$$

then the solution is

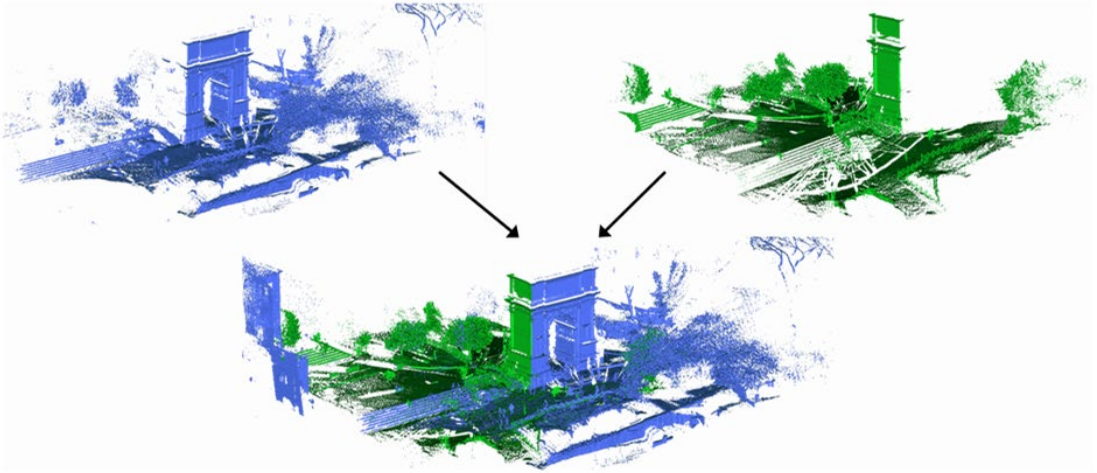
$$R = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(UV^T) \end{pmatrix} V^T$$

Kabsch Algorithm for 3D->3D

- Compute centroids $\bar{A}$ and $\bar{B}$ of the two sets of 3D points.
- Create matrices $A_{3\times n},\ B_{3\times n}$ after subtracting $\bar{A}$ and $\bar{B}$ from all points in the two sets.
- To find R, we must solve: $\underset{R\in SO(3)}{\operatorname{argmin}} ||A - RB||_F = \underset{R\in SO(3)}{\operatorname{argmin}} ||R - AB^T||_F^2$
  - First set $\hat{R} = (AB^T)_{3\times 3}$
  - Then decompose $\hat{R} = \mathrm{U}\Sigma V^T$
  - Set $R = U\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(VU^T) \end{pmatrix} V^T$
- Set $T = \bar{A} - R\bar{B}$

Sidenote: Extensions of 3D Point Set registration: Iterative Closest Point (ICP)

- Widely used for 3D point set registration ***without prior knowledge of correspondences***. Just needs 2 unordered point sets A and B. Jointly finds correspondences and transformation.



- Basic pseudocode:
  - **Step 1:** First, find "closest points" in set A for each point in set B. Treat these as pseudocorrespondences.
  - **Step 2:** Then solve for R, T treating these correspondences as given. E.g. using Kabsch, but several other alternatives.
  - **Step 3:** Transform set B using R, T, and go back to step 1.

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  - **Step 3:** Transform set B using R, T, and go back to step 1.
- **Disadvantages:**
  - Much slower because of having to iterate between correspondences and transformation.
  - Requires a good initialization of R and T.

PnP produces non-unique solutions $(n > 3)$?

- Recall that P3P Step 1 produced non-unique solutions for the distances d_i.
- But if we have $n > 3$ point correspondences, we only need Step 2. This is the **"Perspective-N-Point" problem, or simply PnP.**

Direct solution for PnP ($n > 3$): Steps

Again, first switch to calibrated coordinates:

Pose from N points in space given intrinsic parameters K and correspondences $(X_i, Y_i, Z_i, x_i, y_i)_{i=1...N}$
where

$$\begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} \sim K^{-1} \begin{pmatrix} u_i \\ v_i \\ 1 \end{pmatrix}$$

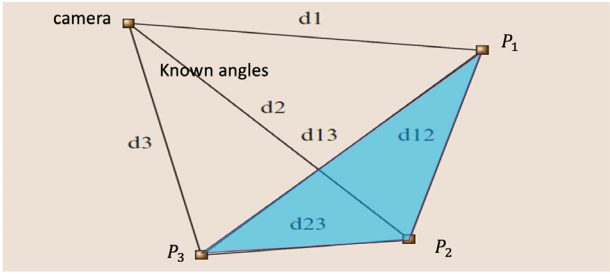
where u_i, v_i are pixel coordinates.

Direct solution for PnP: Steps

Given $(X_i, Y_i, Z_i, x_i, y_i)_{i=1\ldots N}$
find R, T such that

$$\lambda_i \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} = R \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} + T$$

P3P from Pixels



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Identical to the stage we reached with P3P, shown above.

But now, we are after a *direct* solution.

No need to solve explicitly for depths as we did in P3P, so need to find unit vectors etc.

Direct solution for PnP: Steps

Instead substitute λ and get 2 linear equations per point correspondence

$$\lambda_i \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} = R \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} + T \quad \Rightarrow \quad \begin{aligned} x_i &= \frac{r_{11}X_i + r_{12}Y_i + r_{13}Z_i + T_1}{r_{31}X_i + r_{32}Y_i + r_{33}Z_i + T_3} \\ y_i &= \frac{r_{21}X_i + r_{22}Y_i + r_{23}Z_i + T_2}{r_{31}X_i + r_{32}Y_i + r_{33}Z_i + T_3} \end{aligned}$$

Get linear equations by cross-multiplying ...

$$A_{2n\times 12}\boldsymbol{x}_{12} = 0,$$

where $\boldsymbol{x}$ is a vector of all the unknowns including R and T.

Need at least 11 equations = at least 6 point correspondences to solve.
(just like for homography $A_{2n\times 9}\boldsymbol{h}_{9\times 1} = 0$, we needed at least 4 point correspondences = 8 equations.)

Q: Why do we need more points for the direct method, when P3P was able to work with 3 points?

We meet Procrustes and Kabsch yet again

Again, just solving the linear system won't give good rotations. So full steps:

1. Solve $A_{2n\times 12}\boldsymbol{x}_{12} = 0$ from $n \geq 6$ correspondences.
2. Then, assemble the rotation matrix $\hat{R}$ from the solution for $\boldsymbol{x}$.
3. Then, find the closest valid rotation matrix by Kabsch!

Decompose $\hat{R} = \mathrm{U}\Sigma V^T$

Then, set $R = U\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(VU^T) \end{pmatrix} V^T$

4. May now need to adjust translation T to be consistent with new R, e.g., by solving $A_{2n\times 12}\boldsymbol{x}_{12} = 0$, for only t_1, t_2, t_3 (elements of $\boldsymbol{x}$), holding R fixed.