CIS 5800

Machine Perception

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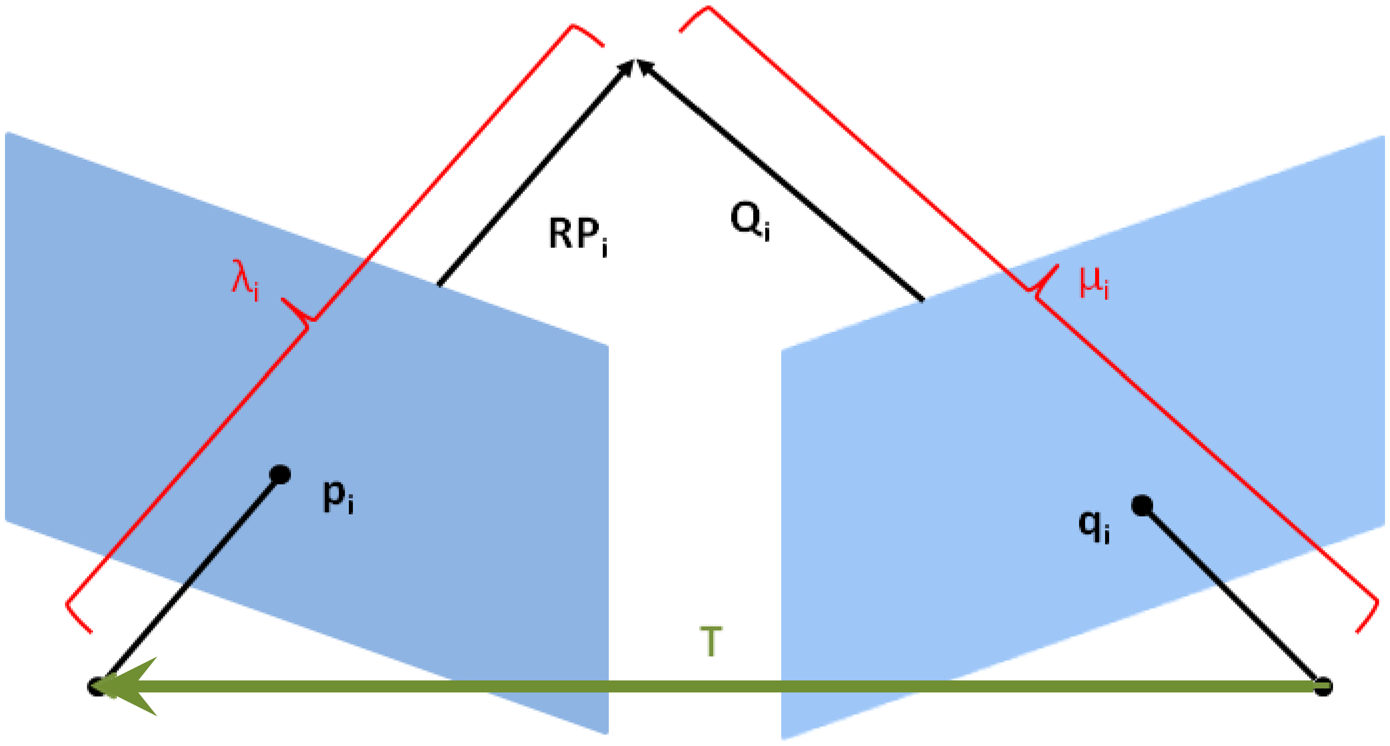
Recap: Two Calibrated Views of the Same 3D Scene

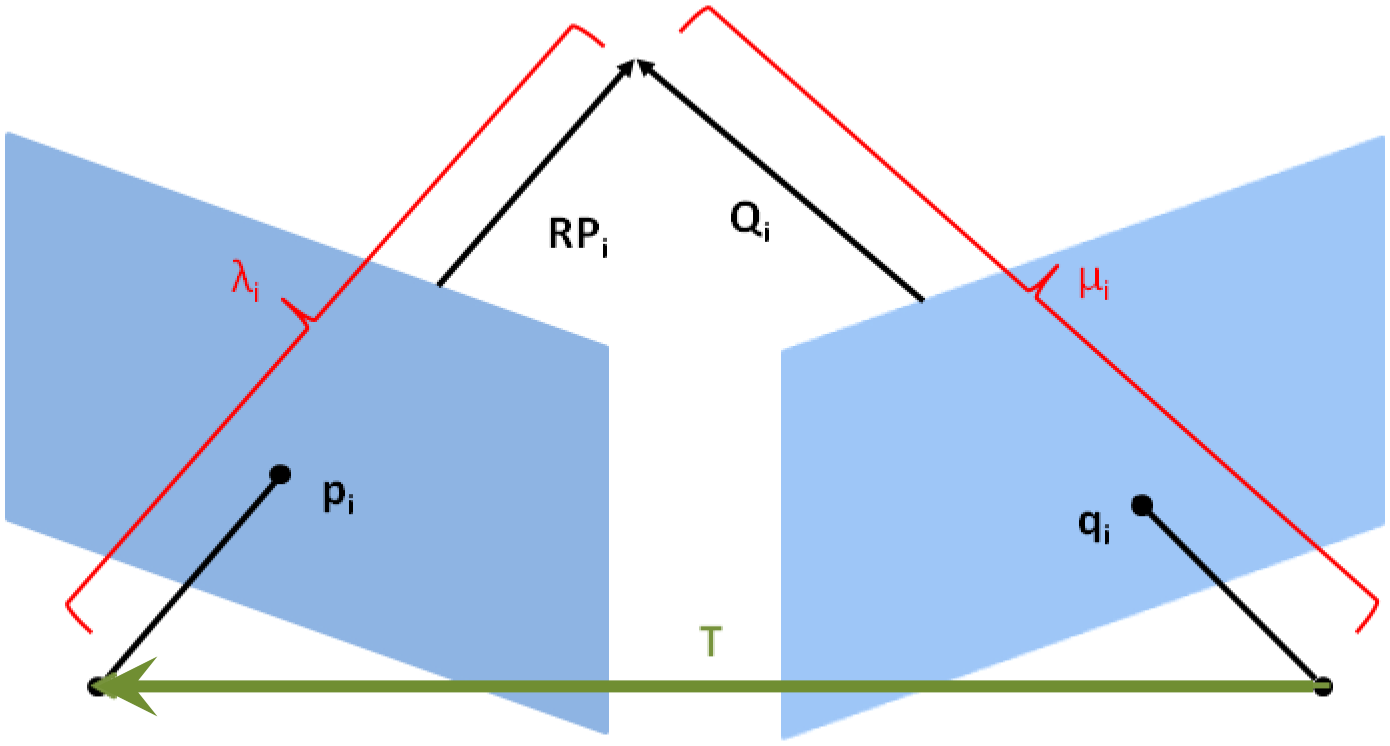


$$R(\lambda p) + T \ = \mu q$$

Given 2D correspondences (p,q)

Find motion R, T and depths λ, μ.

Recap: "Epipolar Constraints" Between Two Views of a Scene



We can eliminate the depths from $R(\lambda p) + T = \mu q$ and obtain the epipolar constraint:

$$\boldsymbol{q}_i^T (T \times R\boldsymbol{p}_i) = 0$$

Recap: The Essential Matrix E

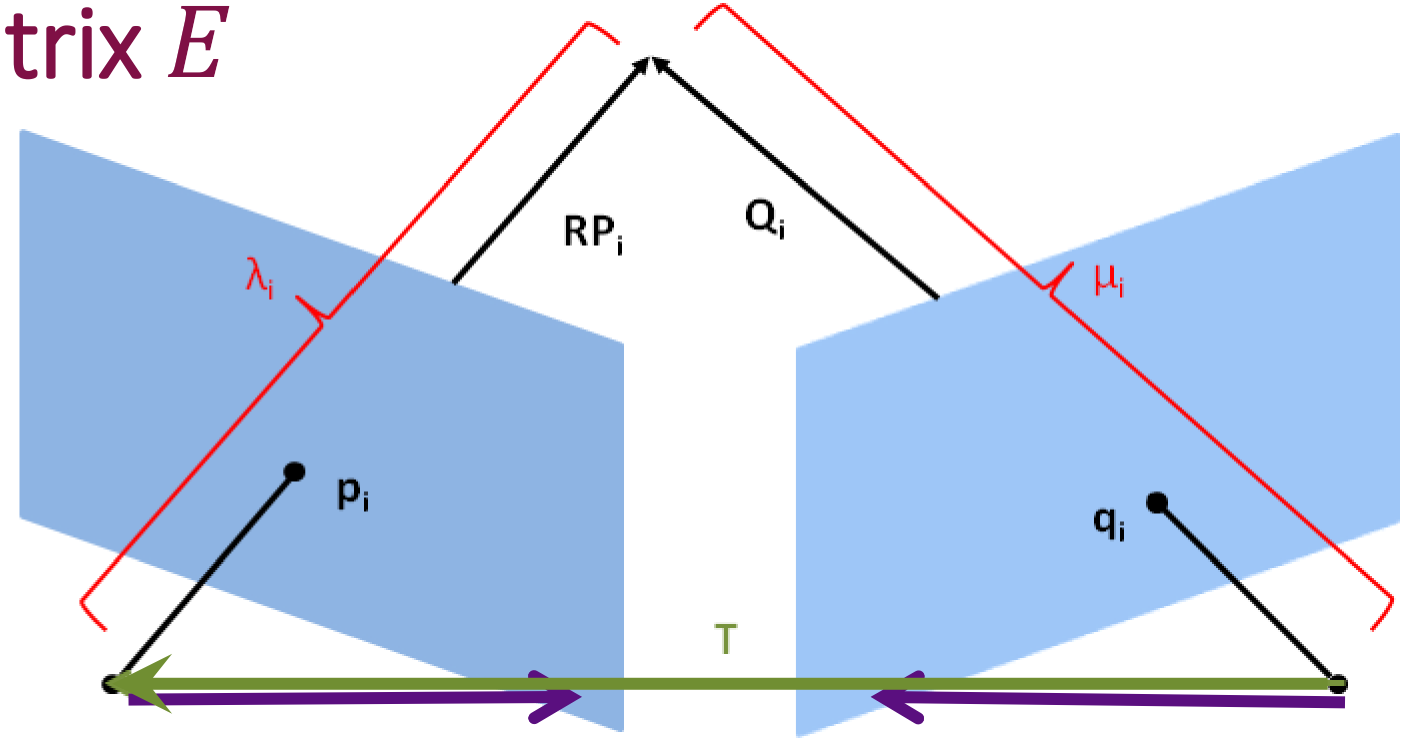
We had: $\boldsymbol{q}_i^T (T \times R\boldsymbol{p}_i) = 0$

$$\Rightarrow \boldsymbol{q}_i^T (\hat{T}R)\ \boldsymbol{p}_i = 0$$

Renaming $E = (\hat{T}R)$:

$$\boldsymbol{q}_i^T E\ \boldsymbol{p}_i = 0$$

"Essential matrix"



$e_p \sim -R^T T$ $\qquad e_q \sim T$

Now linear in the new unknowns $E_{3\times3}$! But will need to recover $T_{3\times1}$, $R_{3\times3}$ later.

Recap: 8-Point Algorithm

- Recall that each correspondence gives us one linear equation in the unknowns E

Is this really linear in E?

$q_i^T E_{3\times3} p_i = 0$ is a single equation that is linear in the elements of E
Can write this out explicitly as below.

If

$$E = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}$$

then epipolar constraint can be rewritten as

$$q^T \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = q^T \left(p_x e_1 + p_y e_2 + p_z e_3 \right)$$

$$= \boxed{\begin{pmatrix} p_x q^T & p_y q^T & p_z q^T \end{pmatrix}} \boxed{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = 0$$

"$a_{1\times9}$" "$E'_{9\times1}$"

This equation is linear

Longuet-Higgins 1981

Recap: 8-Point Algorithm

Let $\vec{a} = \begin{pmatrix} p_x q^T & p_y q^T & p_z q^T \end{pmatrix}$

$$\begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix} E' = 0$$

One row per point correspondence

$$n \times 9$$

where a_i is the known 1 x 9 vector of image points and E' is the essential matrix re-organized into a 9 x 1 column vector.

E' has to be in the null-space of $\begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix}$.

Does this remind you of something?
Hint: 4-Point Collineations, PnP, …

Solution: As before, set E' to the last right singular vector of $A_{n\times 9}$

Longuet-Higgins 1981

Recap: After solving for E, not Quite Done Yet!

$E = \hat{T}R$ has fewer than 8 DOF. T has 3 DOF $(+3)$, R has 3 DOF $(+3)$, and E is scale invariant (-1), so total 5 DOF. **So not any 3x3 matrix is a valid essential matrix.**

- **Problem:** **Given the above, how to ensure that the estimated $\boldsymbol{E}$ is a valid essential matrix?**
- **Problem:** **How to decompose $\boldsymbol{E}$ into the $\hat{\boldsymbol{T}}, \boldsymbol{R}$ required in SfM?**

*https://tutorial.math.lamar.edu/classes/calciii/quadricsurfaces.aspx

Review:

- Singular Value Decomposition (SVD)
- Eigenvalues

Constructing Valid Essential Matrices and Decomposing Them

Necessary and sufficient condition: E is essential **iff** $\sigma_1(E) = \sigma_2(E) \neq 0$ and $\sigma_3(E) = 0$.

Part 1: Proving 'necessary' ("If E is essential, then …") will tell us about properties of essential matrices, so we can correct the E matrices from the direct method to become valid.

Part 2: Proving 'sufficient' ("If singular values …, then …") will help us solve R, T from E for a particular pair of cameras.

Proof Sketch Part 1: Singular Values of A Valid Essential Matrix

$$\sigma_1(E) = \sigma_2(E) \neq 0 \text{ and } \sigma_3(E) = 0.$$

a square matrix whose determinant is zero

E is a singular matrix (Because $E = \hat{T}R$, and $\det(\hat{T}) = 0$)

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

So smallest singular value is indeed zero!

For the two other singular values of E, recall that they are square roots of the eigenvalues of EE^T.
Finding eigenvalues => setting $\det(EE^T - \lambda I) = 0$ i.e. "characteristic polynomial"

$$E = \hat{T}R, \Rightarrow E^T = R^T\hat{T}^T$$

$$EE^T = \hat{T}\hat{T}^T$$

$$= -\begin{bmatrix} t_x^2 & t_x t_y & t_x t_z \\ t_x t_y & t_y^2 & t_y t_z \\ t_x t_z & t_y t_z & t_z^2 \end{bmatrix} + \|T\|^2 I$$

Proof Sketch Part 1: Singular Values of A Valid Essential Matrix

$$\sigma_1(E) = \sigma_2(E) \neq 0 \text{ and } \sigma_3(E) = 0.$$

$$EE^T = \widehat{T}\widehat{T}^T$$

$$= -\begin{bmatrix} t_x^2 & t_x t_y & t_x t_z \\ t_x t_y & t_y^2 & t_y t_z \\ t_x t_z & t_y t_z & t_z^2 \end{bmatrix} + \|T\|^2 I$$

If we solve the characteristic polynomial $\det(EE^T - \lambda I) = 0$ we will find two eigenvalues both equal to $\|T\|^2$. (Bonus exercise: try this out by hand)

So, $\sigma_1(E) = \sigma_2(E) = ||T||$ and $\sigma_3(E) = 0$

Side note: the 'third singular vector' of E (null vector because $\sigma_3 = 0$) is nothing but the translation vector T because:

$$EE^T T = \hat{T}\hat{T}^T T = T \times (-T \times T) = 0!$$

So we already know T in terms of E! (the 3rd left singular vector of E)

Proof Sketch Part 1: Singular Values of A Valid Essential Matrix

Hence, we have proved that **if a matrix is essential, namely, can be decomposed as the product of an antisymmetric $\hat{T}$ and a special orthogonal R then its singular values are $\sigma_1 = \sigma_2 > 0$ and $\sigma_3 = 0$.**

$$\sigma_1(E) = \sigma_2(E) \neq 0 \text{ and } \sigma_3(E) = 0.$$

Condition A

Utility: Having obtained an initial estimate E through direct solution of >=8 epipolar constraints, we may enforce "condition A" above on it, by:

1. Compute SVD $E = U \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} V^T$

2. Then, set $E_{new} = U \begin{bmatrix} (\sigma_1+\sigma_2)/2 & & \\ & (\sigma_1+\sigma_2)/2 & \\ & & 0 \end{bmatrix} V^T$

This satisfies $\underset{E_{new}}{\text{argmin}} \, ||E_{new} - E||_F^2$ s.t. E_{new} meets "condition A".

Since E is scale-invariant, optionally, can also just set Σ to $diag(1,1,0)$

We don't yet know how to get $\boldsymbol{R}, \boldsymbol{T}$ from $\boldsymbol{E}$

Proof Part 2: Construction of E as $\hat{T}R$

Necessary and sufficient condition: E is essential **iff** $\sigma_1(E) = \sigma_2(E) \neq 0$ and $\sigma_3(E) = 0$.

We have to prove the sufficient condition:

If the singular values of a matrix are are $\sigma_1 = \sigma_2 > 0$ and $\sigma_3 = 0$ then the matrix can be decomposed into the product of an antisymmetric $\hat{T}$ and a special orthogonal R.

i.e. valid rotation matrix R, orthonormal with determinant +1 (right-handed coordinate system)

Proof Part 2: Construction of E as $\hat{T}R$

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Consider the simplest matrix satisfying $\sigma_1 = \sigma_2$ and $\sigma_3 = 0$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It indeed can be decomposed into antisymmetric / skew and rotation, e.g.:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#1

skew-symmetric $\widehat{T_{-z}}$ x rotation $R_{z,\pi/2}$

$T_{-z} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ and rotation by 90° about z axis

Proof Part 2: Construction of E as $\hat{T}R$

If the singular values of a matrix are are $\sigma_1 = \sigma_2 > 0$ and $\sigma_3 = 0$ then the matrix can be decomposed into the product of an antisymmetric $\hat{T}$ and a special orthogonal R.

We need one more piece before we can prove this.

If Q is orthogonal ($Q^T Q = I$), then

$$\widehat{Qa} = Q\hat{a}Q^T$$

#2

Proof: $\widehat{Qa}b = Qa \times b = Q(a \times Q^T b) = Q\hat{a}Q^T b.$

It doesn't matter whether you rotate first and then take cross-product or the other way around. i.e., For a rotation R, $R(\boldsymbol{a} \times \boldsymbol{b}) = R\boldsymbol{a} \times R\boldsymbol{b}$

Now let's put #1 and #2 together

Proof Part 2: Construction of E as $\hat{T}R$

For a general essential matrix E that has two equal singular values and one singular value 0, we can write:

$$E = U \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

$$= \sigma U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

$$= \sigma U \widehat{T_{-z}} R_{z,\pi/2} V^T$$

$$= \sigma U \widehat{T_{-z}} U^T U R_{z,\pi/2} V^T$$

$$= \sigma \, (\widehat{U T_{-z}}) \left(U R_{z,\pi/2} V^T \right)$$

#1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

skew-symmetric $\widehat{T_{-z}}$ x rotation $R_{z,\pi/2}$

$$U^T U = I$$

#2

If Q is orthogonal ($Q^T Q = I$), then

$$\widehat{Qa} = Q \hat{a} Q^T$$

Proof Part 2: Construction of E as $\hat{T}R$

For a general essential matrix E, we can write it as the product of a skew-symmetric matrix, and an orthogonal matrix:

$$E = \sigma\widehat{(UT_{-z})}\left(UR_{z,\pi/2}V^T\right)$$

Skew symmetric, since it represents cross product with UT_{-z} = the last column of U= last left singular vector of E (w. flipped sign)

T = last left singular vector of E (w. flipped sign*)

*We'll soon see the sign flipping isn't a big deal

orthogonal, since product of orthogonal matrices

R

Q: Is this the *only* way to decompose $E = \hat{T}R$?

No!

Uniqueness of SfM solution?

$E = \sigma(\widehat{UT_{-z}})\,(UR_{z,\pi/2}V^T)$

We used the following decomposition:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

skew-symmetric $\widehat{T_{-z}}$ x rotation $R_{z,+\pi/2}$

But what if we had instead used the equally valid:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

skew-symmetric $\widehat{T_{+z}}$ x rotation $R_{z,-\pi/2}$

We would have ended up with a different estimate of $\hat{T}, R$:

- $\hat{T} = \sigma\,(\widehat{UT_{+z}})$
- $R = UR_{z,-\pi/2}V^T$

Uniqueness of SfM solution?

- So, for a given E matrix, there are two possible decompositions $\hat{T}, R$

- But this is not all. If E is a valid solution to $q^T E p = 0$, then so is $-E$!
  - And $-E$ induces its own two decompositions into $\hat{T}, R$:
    - $E = \sigma\,(\widehat{UT_{-z}})\left(UR_{z,+\pi/2}V^T\right) \Rightarrow -E = \sigma(\widehat{UT_{+z}})(UR_{z,+\pi/2}V^T)$, and
    - $E = \sigma\,(\widehat{UT_{+z}})\left(UR_{z,-\pi/2}V^T\right) \Rightarrow -E = \sigma(\widehat{UT_{-z}})(UR_{z,-\pi/2}V^T)$

The Four Possible R, T decompositions of $\pm E$

If $E = U\Sigma V^T = U \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$, there are four solutions for the pair $(\hat{T}, R)$

Note: Both R matrices are not guaranteed to have determinant +1. Could be -1. But at least one with +1.

$$\pm E \quad = \quad \hat{T} \quad R$$

$$\sigma(\widehat{UT_{-z}}) \; (UR_{z,+\pi/2}V^T)$$
$$\sigma(\widehat{UT_{+z}}) \; (UR_{z,+\pi/2}V^T)$$
$$\sigma(\widehat{UT_{+z}}) \; (UR_{z,-\pi/2}V^T)$$
$$\sigma(\widehat{UT_{-z}}) \; (UR_{z,-\pi/2}V^T)$$

Can get rid of the scale σ, no harm done.

Last left singular vector of E, with either + or - sign

Can disambiguate by enforcing positive depths for all points (most points if noisy) in both cameras.

But how to get depths?

Computing depths λ_i, μ_i through "triangulation"

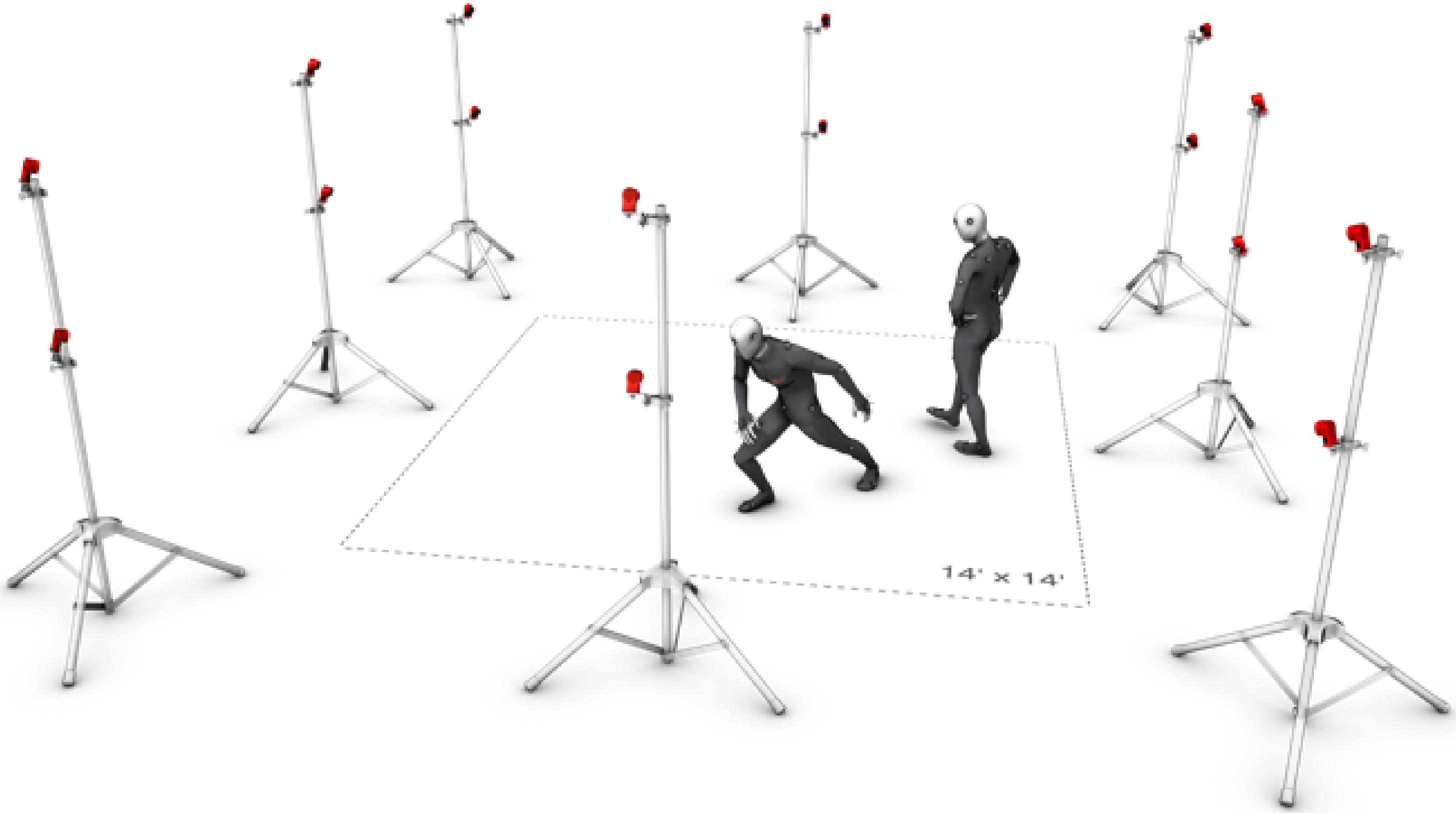
Triangulation is possible if we have computed R and T but again up to a scale factor. Set $\|T\| = 1$:

$$\underbrace{(q_i - Rp_i)}_{3\times 2} \underbrace{\begin{pmatrix} \mu_i \\ \lambda_i \end{pmatrix}}_{2\times 1} = \underbrace{T}_{3\times 1}$$

There are then 3 equations with 2 unknowns λ_i and μ_i for each point.

Solve with pseudo-inverse.

Triangulation is independently useful! E.g. MoCap



Optitrack

The full two-view 8-point algorithm

Direct solution of E

❶ Build the homogeneous linear system by stacking epipolar constraints $q_i^T(T \times Rp_i) = 0,\ i = 1, \ldots, 8$:

$$\underbrace{\begin{bmatrix} \vdots \\ (q_i \otimes p_i)^T \\ \vdots \end{bmatrix}}_{A\ (8\times 9)} \begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix}$$

❷ Let $\begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix}$ be the nullspace of A (if $\sigma_8 \approx 0$ give up)

The full two-view 8-point algorithm

Make E valid

3. $\begin{bmatrix} e_1' & e_2' & e_3' \end{bmatrix} = U\text{diag}\,(\sigma_1' \; \sigma_2' \; \sigma_3')V^T$. Then use the following estimate of the essential matrix:

$$E = U\text{diag}\left(\frac{\sigma_1' + \sigma_2'}{2}, \frac{\sigma_1' + \sigma_2'}{2}, 0\right) V^T$$

Decompose Into T, R

4. $T = \pm \hat{u}_3 \quad R = UR_{Z,\pi/2}V^T$ or $R = UR_{Z,-\pi/2}V^T$
5. Try all four pairs (T, R) to check if reconstructed points are **in front** of the cameras $\boxed{\lambda q = \mu R p + T}$ give $\lambda, \mu > 0$.